The Mathieu group M_{24}
Classification of finite simple groups

Pure Mathematics and Mathematical Logic

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M_{24} The Mathieu group with order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ i, 1, 4, 31, 32

$\mathscr{C}$ The subspace of $\mathcal{P}(\Omega)$, $\mathscr{C} = \{X \in \mathcal{P}(\Omega) : |X| \geqslant 8\}$ iii, 7, 9, 10, 12, 16

M_{11} The Mathieu group with order $2^4 \cdot 3^2 \cdot 5 \cdot 11$ 1

M_{12} The Mathieu group with order $2^6 \cdot 3^3 \cdot 5 \cdot 11$ 1

M_{22} The Mathieu group with order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ 1

M_{23} The Mathieu group with order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ 1

$\mathcal{P}(\Omega)$ The power set of Ω 6

$\mathscr{C}_8$ The Steiner system $S(5, 8, 24)$, there are $\binom{24}{5}/\binom{8}{5} = 759$ octads 7, 12, 16, 24, 25, 31

§ I Introduction

The Mathieu groups are M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, which are one family of the 26 sporadic finite simple groups. They were discovered by the French mathematician Emile Mathieu, see table I.1. Furthermore, the first expression of simplicity and uniqueness of Mathieu groups was in 1930s in a paper by Witt, and a Steiner system was described in that paper. Now, we are normally using the system to describe these groups. The largest Mathieu group is M_{24} which is 5-transitive of 24-point, and it could be defined as a group of preserved permutations of Steiner system $S(5, 8, 24)$. Notice that, some of M_{24}'s simple subgroups are M_{23}, M_{22}, M_{12} and M_{11}. This project is started by introducing some basic facts, moving to the definition of Steiner system and some of its properties, following by discussing the Steiner system $S(5, 8, 24)$, then presenting the Miracle Octad Generator (**MOG**). In the end, we introduc the concept of M_{24}. The main two sources for this project are (*An introduction to Steiner systems* by M. Grannell and T. Griggs, [GrGr]) and (*A new combinatorial approach to* M_{24} by R. Curtis, [Cur]).

Group	Order	Discovered by	Date
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	Mathieu	1873
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	Mathieu	1873
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	Mathieu	1873
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	Mathieu	1873
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	Mathieu	1873

Table I.1: The Mathieu groups

§ II PRELIMINARIES

II.1 Background

Definition II.1.1. $GF(2)$ is a *finite field* with 2 elements $\{0, 1\}$.

Definition II.1.2. A set that contains four elements is called a *tetrad*.

Definition II.1.3. Let V is a non-empty set and K is a field. Then V is called a *vector space* over K if V is an abelian group under addition and it is closed under a scalar multiplication, let $\lambda, \beta \in F, v, v_1, v_2 \in V$, then

1. $(\lambda + \beta)v = \lambda v + \beta v$,
2. $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$,
3. $(\lambda\beta)v = \lambda(\beta v)$, and
4. $1v = v$.

Definition II.1.4. Let G be a group and $N \geqslant G$,then N is a *normal subgroup* if and only if for all $g \in G$ we have $N^g = N$.

Definition II.1.5. A non-trivial group G is called a *simple* group if it has no proper non-trivial normal subgroups.

Definition II.1.6. An *action* of a group G on a non-empty set Ω is a binary operation $* : \Omega \times G \to \Omega$, such that for all $\alpha \in \Omega$, $\alpha * 1 = \alpha$ and $(\alpha * g) * h = \alpha * (gh)$ for all $g, h \in G$. The degree of action G on ω is the cardinality of Ω.

Definition II.1.7. Let G act on a set Ω and $\alpha \in \Omega$. Then $\alpha G = \{\alpha g \mid g \in G\} \subseteq \Omega$ is called *the orbit* of α under G.

Definition II.1.8. Let G be a group acting on a non-empty set Ω.

1. If $\alpha \in \Omega$ then $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is called *the stabiliser* of α.

2. If G has only one orbit, i.e. $\alpha G = \Omega$ for $\alpha \in \Omega$ then we say that G acts *transitively* on Ω.

3. A group G acts k*-transitively* precisely if for any two sequences of k distinct points from Ω, say $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\beta_1, \beta_2, \ldots, \beta_k)$ there is a group element $g \in G$ such that $\alpha_i g = \beta_i$ for each $i = 1, \ldots, k$.

4. If G be k-transitive, and distinct elements $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Omega$, $g_1, g_2 \in G$ satisfy $\alpha_i g_1 = \alpha_i g_2$ for all $i = 1, \ldots, k$ and $g_1 = g_2$, then we say that G acts *sharply* k*-transitive* on Ω.

Definition II.1.9. Let X, Y be sets, then *symmetric difference* defines as $X + Y = Z$, where

$$Z = \{x \mid (x \in X \wedge x \notin Y) \vee (x \notin X \wedge x \in Y)\} \quad \text{or} \quad Z = (X \backslash Y) \bigcup (Y \backslash X).$$

Definition II.1.10. Let G and H be groups $\phi : G \to H$ is a group homomorphism if

$$(g_1 g_2)\phi = (g_1)\phi(g_2)\phi \quad \text{for all } g_1, g_2 \in G.$$

§ III The Mathieu group M_{24}

III.1 Steiner system

Definition III.1.1. A *Steiner system* $S(t, k, v)$ is a set of k-element subsets of a base set which is a set of v elements and any t-element subset of the base set appears in precisely one of the k-element subsets which are called *blocks*.

Theorem III.1.2. *If there exists an $S(t, k, v)$ then there exists an $S(t-1, k-1, v-1)$.*

Proof. Suppose $S(t, k, v)$ exists, Ω is a base set and α is a fixed element in Ω. Then remove all the blocks which do not contain α, so the remaining blocks contain α and $t-1$ elements which appear precisely once in a block. By removing α from Ω and the blocks we obtain that $|\Omega \backslash \{\alpha\}| = v-1$, and the size of blocks is $k-1$. Hence, this is an $S(t-1, k-1, v-1)$. □

Theorem III.1.3. *If there exists an $S(t, k, v)$. Then $\binom{k}{t}$ divides $\binom{v}{t}$, and the number of blocks is $\binom{v}{t}/\binom{k}{t}$.*

Proof. Suppose $S(t, k, v)$ exists, $|\Omega| = v$ and $X \subseteq \Omega$ with $|X| = k$, then the number of all subsets of size t of X is $\binom{k}{t}$. By assuming there are n sets of size k, and since any t-element lies in only one k-element, this implies that

$$n \times \binom{k}{t} = \binom{v}{t}$$

and n is an integer, hence $\binom{v}{t}/\binom{k}{t}$.

□

Corollary III.1.4. *If there exists an $S(t,k,v)$ then $\binom{k-i}{t-i}$ divides $\binom{v-i}{t-i}$ for each $i = 0,1,2,\ldots,t-1$.*

Proof. Indeed by using Theorem III.1.2 there exists $S(t-i,k-i,v-i)$ for each $i = 0,1,2,\ldots,t-1$. Then apply Theorem III.1.3.

□

Example III.1.5. In a Steiner system $S(2,3,9)$, the base set is Ω which has nine elements and for convenience suppose $\Omega = \{1,2,3,4,5,6,7,8,9\}$ with positive integers. Then our blocks have size 3 with the property that any pair lies in only one block. Notice that, the number of blocks from Theorem III.1.3 is $\binom{9}{2}/\binom{3}{2} = 12$, and the blocks are $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{1,4,7\}$, $\{2,5,8\}$, $\{3,6,9\}$, $\{1,5,9\}$, $\{2,6,7\}$, $\{3,4,8\}$, $\{2,4,9\}$, $\{3,5,7\}$ and $\{1,6,8\}$.

Example III.1.6. In a Steiner system $S(3,4,8)$, the base set is Ω which has eight elements and for convenience suppose $\Omega = \{A,B,C,D,E,F,G,H\}$. Then our blocks have size 4 with property that any 3-element lies in only one block. Notice that, the number of blocks from Theorem III.1.3 is $\binom{8}{3}/\binom{4}{3} = 14$, and the blocks are $\{A,B,C,H\}$, $\{A,D,E,H\}$, $\{A,F,G,H\}$, $\{B,D,F,H\}$, B,E,G,H, $\{C,D,G,H\}$, $\{C,E,F,H\}$, $\{D,E,F,G\}$, $\{B,C,F,G\}$, $\{B,C,D,E\}$, $\{A,C,E,G\}$, $\{A,C,D,F\}$, $\{A,B,D,G\}$ and $\{A,B,E,F\}$.

Corollary III.1.7. *The $S(2,3,7)$ exists. This is clear from Theorem III.1.2 and existence of $S(3,4,8)$.*

Remark III.1.8. It is not necessary there is a Steiner system for any three integers numbers as an example $S(2,3,8)$ is not a Steiner system since $\binom{3}{2}$ does not divide $\binom{8}{2}$ which contradiction with Theorem III.1.3.

III.2 Steiner system $S(5,8,24)$

Definition III.2.1. A *Steiner system* $S(5,8,24)$ is a set of all sets of size 8, which are subsets of a set of size 24 elements, say Ω with property that any subset of size 5 of Ω appears in only one of the 8-element sets which are called octads, i.e.

$$S(5,8,24) = \{B \subseteq \Omega : \forall X \subseteq \Omega\ \exists!\, B \mid X \subseteq B, |B| = 8 \text{ and } |X| = 5\}.$$

Theorem III.2.2. *[Cur, Theorem A] A Steiner system $S(5, 8, 24)$ exists.*

The first claim is that the power-set of 24-element set is a vector space with 24 dimensions over $GF(2)$ and the addition operation is defined by symmetric difference in $\mathcal{P}(\Omega)$.

Proof. Suppose $\Omega = \{a_1, a_2, a_3, a_4, \ldots, a_{24}\}$, and V is a vector space with 24 dimensions over $GF(2)$, which has a standard basis $\{(1, 0, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1)\}$. Moreover, the operation is symmetric difference. Let define the map

$$\phi : \mathcal{P}(\Omega) \to V, \quad \text{by}$$

$$Y \mapsto (i_1, \ldots, i_j, \ldots, i_{24}), \quad \begin{cases} i_j = 1 & \text{if } a_i \in Y, \\ i_j = 0 & \text{if } a_i \notin Y. \end{cases}$$

We need to show that ϕ is surjective, injective and a group homomorphism, let Y, X be subsets of Ω and $(Y)\phi = (i_1, \ldots, i_j, \ldots, i_{24}) = (X)\phi = (k_1, \ldots, k_j, \ldots, k_{24})$. This implies that $i_j = k_j$ for all $j = 1, \ldots, 24$ and $i_j = 1 = k_j$ for some j. Hence, $a_j \in X$ and $a_j \in Y$, thus, $X = Y$. Therefore, ϕ is an injective function. Notice that, since $|V| = |\mathcal{P}(\Omega)| = 224$ and ϕ is an injection, this implies that ϕ is a surjective function. Now, we need to show that ϕ is a group homomorohism, suppose that $X, Y \in \mathcal{P}(\Omega)$ and

$$X + Y = Z = \{a_j \mid a_j \in X \backslash Y \text{ or } a_j \in Y \backslash X\}.$$

Hence, $(X + Y)\phi = (Z)\phi = (t_1, t_2, \ldots, t_j, \ldots, t_{24})$, where $t_j = 1$ if $k_j \neq i_j$ and zero otherwise. This is the sum of

$$(k_1, k_2, \ldots, k_j, \ldots, k_{24}) + (i_1, i_2, \ldots, i_j, \ldots, i_{24}) = (X)\phi + (Y)\phi.$$

Moreover, it is easy to see that $(\lambda X)\phi = \lambda(X)\phi$, where $\lambda \in GF(2) = \{0, 1\}$ and $X \in \mathcal{P}(\Omega)$. Thus, $\mathcal{P}(\Omega) \cong V$. As a result, $\mathcal{P}(\Omega)$ is a vector space over $GF(2)$ with basis $(e_1, e_2, \ldots, e_{24})$, where

$$e_i = \overset{\qquad\quad i^{th}}{\begin{array}{|c|c|c|}\hline & & \\ \hline & x & \\ \hline \end{array}} \mapsto (0,0,\ldots,\overset{i^{th}}{1},\ldots,0),$$

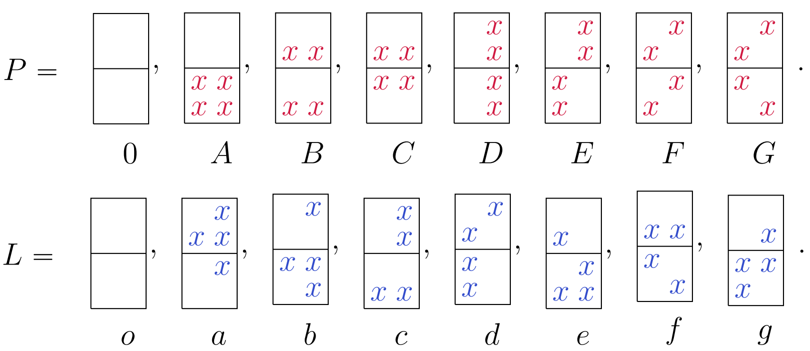
where $i = 1,2,3,\ldots,24$. □

Example III.2.3. If $X = \{a_2, a_3, a_{14}, a_{15}\} =$ then

$$X = \{a_2, a_3, a_{14}, a_{15}\} = \begin{array}{|c|c|c|}\hline x & x & \\ \hline x & x & \\ \hline \end{array} \mapsto (0,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,\ldots,0).$$

The second claim is that we need to produce a subspace $\mathscr{C}$ of $\mathcal{P}(\Omega)$ such that $\mathscr{C} = \{X \in \mathcal{P}(\Omega) : |X| \geqslant 8\}$. Moreover, $\mathscr{C}$ contains 759 octads and the set that contains all these is called $\mathscr{C}_8$ which is $S(5,8,24)$. Notice that, if the Steiner system $S(5,8,24)$ exists, then from Theorem III.1.3 there are $\binom{24}{5}/\binom{8}{5} = 759$ octads.

Proof. Let $\Lambda = \begin{array}{|cc|}\hline x & x \\ x & x \\ \hline x & x \\ x & x \\ \hline \end{array}$ be a set of 8 element and consider $\mathcal{P}(\Lambda)$ as a 8-dimensional vector space over $GF(2)$ as before. Now, suppose that we have any two subspaces of $\mathcal{P}(\Lambda)$, let say P and L which are 3-dimensional, whose members are tetrads, and $P \bigcap L = \varnothing$. Let assume P and L as following:



As you see from table III.1, $(P, *)$ is an abelian group, and it is closed under multiplication by λ, where $\lambda \in GF(2) = \{0, 1\}$. Hence, P is a subspace of $\mathcal{P}(\lambda)$ over $GF(2)$.

$*$	0	A	B	C	D	E	F	G
0	0	A	B	C	D	E	F	G
A	A	0	C	B	E	D	G	F
B	B	C	0	A	F	G	D	E
C	C	B	A	0	G	F	E	D
D	D	E	F	G	0	A	B	C
E	E	D	G	F	A	0	C	B
F	F	G	D	E	B	C	0	A
G	G	F	E	D	C	B	A	0

Table III.1: Group multiplication of elements in P

Similarly, As you see from table III.2, $(L, *)$ is an abelian group, and it is closed under multiplication by λ, where $\lambda \in GF(2) = \{0, 1\}$. Hence, L is also a subspace of $\mathcal{P}(\lambda)$ over $GF(2)$.

$\square$

Remark III.2.4. 1. P is called the *point-space*, and L is called the *line-space.*

2. Intersection of any two members of P (or L) is a subset of size 2, i.e. if $X, Y \in P$ (or L), then $|X \bigcap Y| = 2$.

3. Any three linearly independent of elements of P (or L) can be its basis. For example, $A, B, D \in P$ are linearly independent, thus, $\{A, B, D\}$ can be a basis of P.

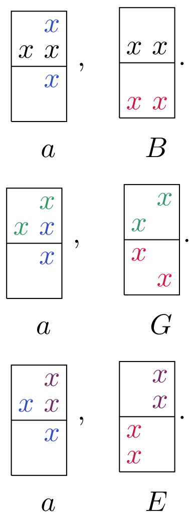
4. For any member of L, let say t, there is a one-to-one correspondence to a 2-dimensional vector space, whose members are from P. This means that for any member of L there are three members of P, which have only two common points with the member of L and

$*$	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	0	f	e	g	c	b	d
b	b	f	0	g	e	d	a	c
c	c	e	g	0	f	a	d	b
d	d	g	e	f	0	b	c	a
e	e	c	d	a	b	0	g	f
f	f	b	a	d	c	g	0	e
g	g	d	c	b	a	f	e	0

Table III.2: Group multiplication of elements in L

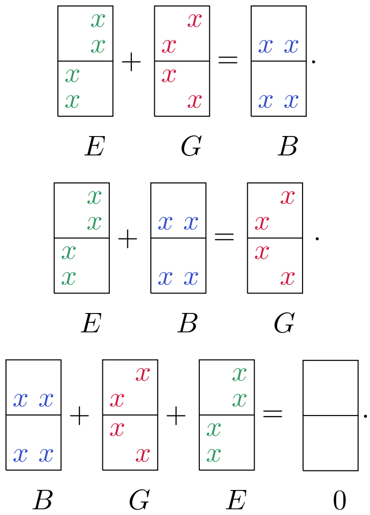
it has dimension two, because it is generated by any two non-zero elements, whereas the third element is their addition, see figure III.2.1.

Example III.2.5. Suppose that $\{0, B, G, E\} \subseteq P$ then it is a 2-dimensional vector space over $GF(2)$. This is the correspondence to a, see figure III.2.1. Firstly, we need to check common points between them

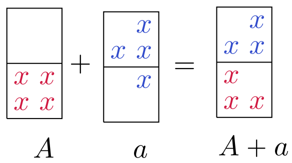


Notice that, there are two common points between a and B, G, E. Movreover, $\{0, E, B, G\}$

is a vector subspace of $\mathscr{C}$, since it is an abelian group, and it is closed under multiplication by a scalar. Notice that, each non-zero element B, G, E has order two and its inverse is the element itself. It is two dimensional since it is generated by any two non-zero members, let say E and B such that $G = E + B$



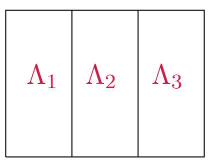
Remark III.2.6. 1. If $|X + t| = 4$ then $X \in t$ and if $|X + t| = 2$ or 6 then $X \notin t$, i.e. $X \in t$, if $|X \bigcap t| = 2$ and $X \notin t$, if $|X \bigcap t| = 1$ or 3. An example



this implies that $A \notin a$.

2. From 1, we notice that any even subset of Λ can be expressed uniquely as $(X + t)$ or $(X' + t)$, where X' is a subset of Λ, and $X + X' = \Lambda$. X' is called the *complement* of X.

Let consider three copies of Λ and define the 12-dimensional space $\mathscr{C}$ of $\mathcal{P}(\Omega)$



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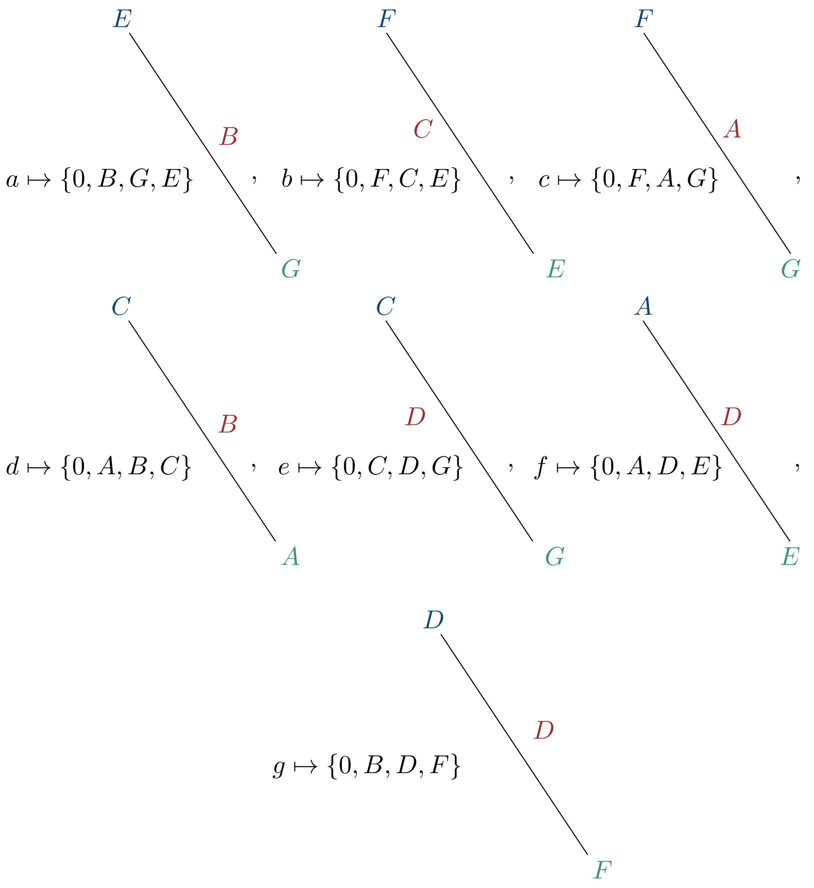


Figure III.2.1: The correspondence map

Such that $\{[(X \text{ or } X')(Y \text{ or } Y')(Z \text{ or } Z')]_t : X, Y, Z \in P, t \in L, X + Y + Z = 0\}$ this set is called $\mathscr{C}$-*set*, where

$$[(X \text{ or } X')(Y \text{ or } Y')(Z \text{ or } Z')]_t = \begin{array}{|c|c|c|} \hline X \text{ or } X' & Y \text{ or } Y' & Z \text{ or } Z' \\ + & + & + \\ t & t & t \\ \hline \end{array}.$$

Example III.2.7. Let $A, B, C \in P$, $A + B + C = 0$ and $e \in L$ then

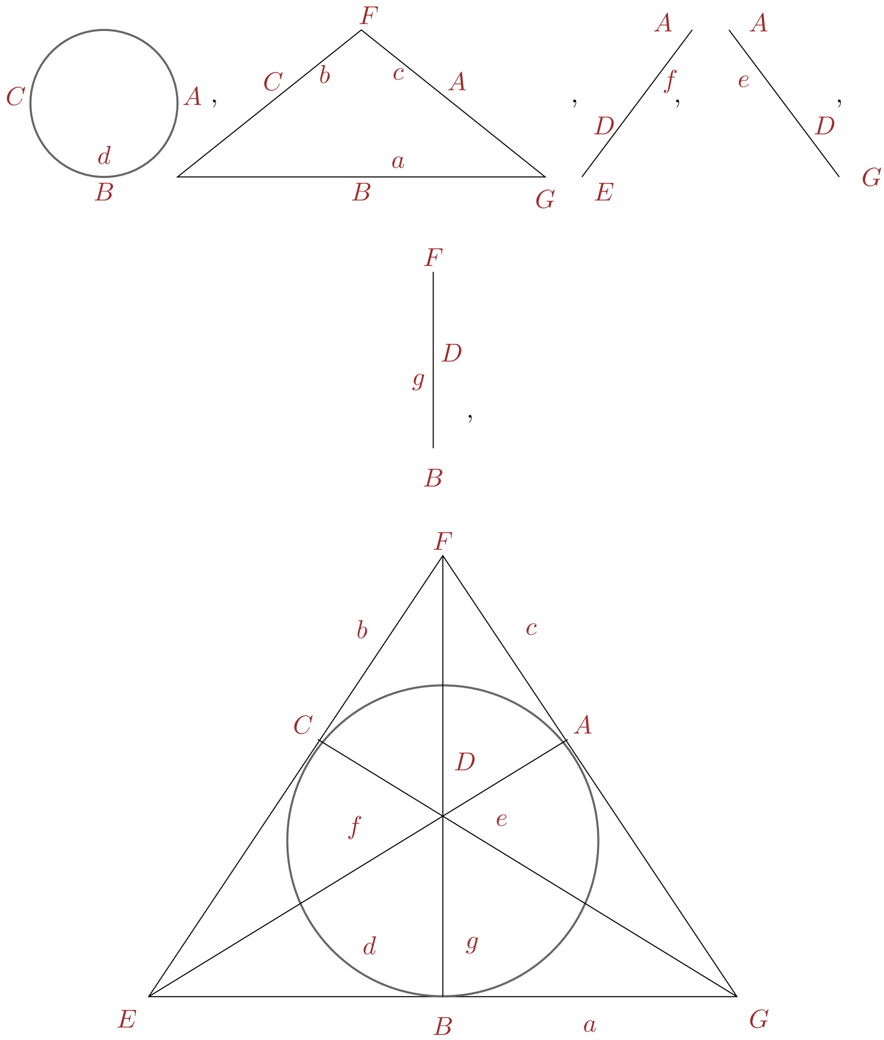


Figure III.2.2: The one-to-one correspondence

$$[A\ B\ C]_e = [A+e, B+e, C+e] = \begin{array}{|c|c|c|} \hline x & \quad x & \quad x \\ \hline x & \quad x & x \\ & & x\ \ x \\ \hline \end{array} .$$

Investigation the size of members of $\mathscr{C}$-set From tables III.3, III.4 and III.5 we can obtain $\mathscr{C}_8$ from $\mathscr{C}$-set by taking shapes which intersect with $(\Lambda_1 + \Lambda_2 + \Lambda_3)$ in 8 points.

Shapes	Description	Intersection with		
		Λ_1	Λ_2	Λ_3
$[0\ 0\ 0]_0$	$\lvert 0\rvert = 0$	0	0	0
$[0\ 0\ 0]_t$	$t \neq 0$	4	4	4
$[0'\ 0\ 0]_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	8	0	0
$[0\ 0'\ 0]_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	0	8	0
$[0\ 0\ 0']_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	0	0	8
$[0'\ 0'\ 0]_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	8	8	0
$[0'\ 0\ 0']_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	8	0	8
$[0\ 0'\ 0']_0$	$\lvert 0\rvert = 0, 0' = \Lambda, \lvert 0'\rvert = 8$	0	8	8
$[0'\ 0'\ 0']_0$	$0' = \Lambda, \lvert 0'\rvert = 8$	8	8	8
$[0'\ 0'\ 0']_t$	$0' = \Lambda, \lvert 0'\rvert = 8, t \neq 0$	4	4	4
$[X\ X\ 0]_0$	$\lvert X\rvert = 4, \lvert 0\rvert = 0$	4	4	0
$[X\ 0\ X]_0$	$\lvert X\rvert = 4, \lvert 0\rvert = 0$	4	0	4
$[0\ X\ X]_0$	$\lvert X\rvert = 4, \lvert 0\rvert = 0$	0	4	4
$[X'\ X'\ 0]_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	0
$[X'\ 0\ X']_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	0	4
$[0\ X'\ X']_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	0	4	4
$[X\ X\ 0']_0$	$\lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	4	8
$[X\ 0'\ X]_0$	$\lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	8	4
$[0'\ X\ X]_0$	$\lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	8	4	4
$[X'\ X'\ 0']_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	4	8
$[X'\ 0'\ X']_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	8	4
$[0'\ X'\ X']_0$	$X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	8	4	4
$[X\ X\ 0]_t$	$X \in t \neq 0, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[X\ 0\ X]_t$	$X \in t \neq 0, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[0\ X\ X]_t$	$X \in t \neq 0, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[X'\ X'\ 0]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[X'\ 0\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[0\ X'\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4

Table III.3: The size of members of $\mathscr{C}$-set I

Shapes	Description	Intersection with		
		Λ_1	Λ_2	Λ_3
$[X\ X'\ 0]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4$	4	4	4
$[X'\ X\ 0]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	4
$[0\ X\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	4
$[0\ X'\ X]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	4
$[X\ 0\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	4
$[X'\ 0\ X]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0\rvert = 0$	4	4	4
$[X\ X\ 0']_t$	$X \in t \neq 0, \lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	4	4
$[X\ 0'\ X]_t$	$X \in t \neq 0, \lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	4	4
$[0'\ X\ X]_t$	$X \in t \neq 0, \lvert X\rvert = 4, 0' = \Lambda, \lvert 0'\rvert = 8$	4	4	4
$[X'\ X'\ 0']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[0'\ X'\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X'\ 0'\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X\ X'\ 0']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X'\ X\ 0']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[0'\ X'\ X]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X\ 0'\ X']_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X'\ 0'\ X]_t$	$X \in t \neq 0, X + X' = \Lambda, \lvert X\rvert = \lvert X'\rvert = 4, \lvert 0'\rvert = 8$	4	4	4
$[X\ X\ 0]_t$	$X \notin t \neq 0, \lvert X\rvert = 2, \lvert X'\rvert = 6, \lvert 0\rvert = 0$	2	2	4
$[X\ 0\ X]_t$	$X \notin t \neq 0, \lvert X\rvert = 2, \lvert X'\rvert = 6, \lvert 0\rvert = 0$	2	4	2
$[0\ X\ X]_t$	$X \notin t \neq 0, \lvert X\rvert = 2, \lvert X'\rvert = 6, \lvert 0\rvert = 0$	4	2	2
$[X'\ X'\ 0]_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6, \lvert 0\rvert = 0$	6	6	4
$[X'\ 0\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	6	4	6
$[0\ X'\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	4	6	6
$[X\ X'\ 0]_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	2	6	4
$[X'\ X\ 0]_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	6	2	4
$[0\ X\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	4	2	6
$[0\ X'\ X]_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	4	6	2
$[X\ 0\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \lvert X\rvert = 2, \lvert X'\rvert = 6$	2	4	6

Table III.4: The size of members of $\mathscr{C}$-set II

Shapes	Description	Intersection with		
		Λ_1	Λ_2	Λ_3
$[X'\ 0\ X]_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6$	6	4	2
$[X\ X\ 0']_t$	$X \notin t \neq 0, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	2	2	4
$[X\ 0'\ X]_t$	$X \notin t \neq 0, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	2	4	2
$[0'\ X\ X]_t$	$X \notin t \neq 0, \vert X\vert = 2, \vert X'\vert = 6$	4	2	2
$[X'\ X'\ 0']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	6	6	4
$[X'\ 0'\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	6	4	6
$[0'\ X'\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	4	6	6
$[X\ X'\ 0']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	2	6	4
$[X'\ X\ 0']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	6	2	4
$[0'\ X\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	4	2	6
$[0'\ X'\ X]_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	4	6	2
$[X\ 0'\ X']_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	2	4	6
$[X'\ 0'\ X]_t$	$X \notin t \neq 0, X + X' = \Lambda, \vert X\vert = 2, \vert X'\vert = 6, \vert 0'\vert = 8$	6	4	2
$[X\ Y\ Z]_0$	$X + Y + Z = 0, \vert X\vert = \vert Y\vert = \vert Z\vert = 4$	4	4	4
$[X'\ Y'\ Z']_0$	$X + Y + Z = 0, \vert X'\vert = \vert Y'\vert = \vert Z'\vert = 4$	4	4	4
$[X\ Y\ Z]_t$	$X, Y \in t, X + Y + Z = 0, \vert X\vert = \vert Y\vert = \vert Z\vert = 4$	4	4	4
$[X'\ Y'\ Z']_t$	$X, Y, Z \in t, X + Y + Z = 0, \vert X\vert = \vert Y\vert = \vert Z\vert = 4$	4	4	4
$[X\ Y\ Z]_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X\vert = 4, \vert Y\vert = \vert Z\vert = 2$	4	2	2
$[X\ Y'\ Z]_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X\vert = 4, \vert Y'\vert = 2, \vert Z\vert = 6$	4	6	2
$[X\ Y\ Z']_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X\vert = 4, \vert Y\vert = 2, \vert Z'\vert = 6$	4	2	6
$[X\ Y'\ Z']_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X\vert = 4, \vert Y'\vert = \vert Z'\vert = 6$	4	6	6
$[X'\ Y\ Z]_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X'\vert = 4, \vert Y\vert = \vert Z\vert = 2$	4	2	2
$[X'\ Y'\ Z]_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X'\vert = 4, \vert Y'\vert = 6, \vert Z\vert = 2$	4	6	2
$[X'\ Y\ Z']_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X'\vert = 4, \vert Y\vert = 2, \vert Z'\vert = 6$	4	2	6
$[X'\ Y'\ Z']_t$	$X \in t, Y, Z \notin t, X + Y + Z = 0, \vert X'\vert = 4, \vert Y'\vert = \vert Z'\vert = 6$	4	6	6

Table III.5: The size of members of $\mathscr{C}$-set III

Case	Shapes	The number of Octads
1	$[0'\ 0\ 0]_0$, $[0\ 0'\ 0]_0$, $[0\ 0\ 0']_0$	$1 \times 3 = 3$
2	$[X\ X\ 0]_0$, $[X\ 0\ X]_0$, $[0\ X\ X]_0$, $[X'\ X'\ 0]_0$, $[X'\ 0\ X']_0$, $[0\ X'\ X']_0$	$7 \times 12 = 84$
2	$[X\ X'\ 0]_0$, $[X\ 0\ X']_0$, $[0\ X\ X']_0$, $[X'\ 0\ X]_0$, $[X'\ 0\ X]_0$, $[0\ X'\ X]_0$	$7 \times 12 = 84$
3	$[X\ X'\ 0]_t$, $[X\ 0\ X']_t$, $[0\ X\ X']_t$, $[X'\ 0\ X]_t$, $[X'\ 0\ X]_t$, $[0\ X'\ X]_t$	$7 \times 4 \times 6 = 168$
4	$[X\ Y\ Z]_t$, $[X\ Z\ Y]_t$, $[X'\ Y\ Z]_t$, $[X'\ Z\ Y]_t$	$7 \times 3 \times 3 \times 2 \times 4 = 504$

Table III.6: Count the Octads

Table III.6 shows all possibilities for the octads shape. In particular, the first case shows all possibilities for ordering $0, 0'$ and $t = 0$, which are three possibilities. The second case presents all the possibilities of ordering $X, X', 0$ and $t = 0$, which are 12 shapes times the number of ways of choosing $X \in P\backslash\{0\} = \{A, B, C, D, E, F, G\}$ which is 7. The third case shows all the possibilities of ordering $X, X', 0$ and $X \notin t \neq 0$, which are 6 shapes. And from figure III.2.2, we can find out there are four possibilities to choose $t \in L\backslash\{0\} = \{a, b, c, d, e, f, g\}$, where $X \notin t$, $(|X + t| \neq 4)$, so it is $4 \times 6 \times 7$. The fourth case shows all the possibilities of ordering X, X', Y, Z and $X \in t \neq 0$, $Y \notin t$, $Z \notin t$ and $X + Y + Z = 0$, which are four shapes. Moreover, From figure III.2.2, there are three possibilities to choose $t \in L\backslash\{0\} = \{a, b, c, d, e, f, g\}$, where $X \in t$, $X \in P\backslash\{0\}$. And there are three choices to choose $Y \in P\backslash\{0, X\} = \{A, B, C, D, E, F, G\}$ and there are two choices to choose $Z \in P\backslash\{0, X, Y\} = \{A, B, C, D, E, F, G\}$. Hence, $7 \times 3 \times 3 \times 2 \times 4$. Therefore, $|\mathscr{C}_8| = 3 + 84 + 168 + 504 = 759$ octads. Furthermore, it is satisfied that any 5-element lies in only one Octad of $\mathscr{C}_8 \subseteq \mathscr{C}$. (If it is not, then there exists $X, Y \in \mathscr{C}_8$ such that $|X| = 8$, $|Y| = 8$ and $|X \bigcap Y| = 5$, but this is impossible since that $|X + Y| \leqslant 6$, and $X + Y \notin \mathscr{C}$ which is a contradiction the fact that $\mathscr{C}$ is a vector space.) Hence, $S(5, 8, 24) = \mathscr{C}_8$, and $|S(5, 8, 24)| = 759$.

Definition III.2.8. If X is an octad which not equal to Λ_1, Λ_2 or Λ_3, then $|X \bigcap \Lambda_i| = 4$ for some $i = 1, 2, 3$. We called Λ_i a *heavy brick* for X. In this case $|X \bigcap (\Lambda_j + \Lambda_k)| = 4$ for $\{i, j, k\} = \{1, 2, 3\}$ and we called $X \bigcap (\Lambda_j + \Lambda_k)$ a *square tetrad*.

Example III.2.9. As you see in Λ_1' is $A' + 0$ and in Λ_2' is $A + 0$, whereas Λ_3' is empty. Since $|\Lambda_1'| = |\Lambda_2'| = 4$ then either Λ_1' or Λ_2' is a heavy brick and a square tetrad

$$[A'\ A\ 0]_0 = \begin{array}{c} \begin{array}{ccc} \Lambda_1' & \Lambda_2' & \Lambda_3' \end{array} \\ \begin{array}{|c|c|c|} \hline \begin{smallmatrix} x & x \\ x & x \end{smallmatrix} & & \\ \hline & \begin{smallmatrix} x & x \\ x & x \end{smallmatrix} & \\ \hline \end{array} \end{array}.$$

Example III.2.10. As you see in Λ_1' is $0' + a = \Lambda + a = a'$ and $|\Lambda_1'| = 4$, whereas $|\Lambda_2 + \Lambda_3'| = 4$. This implies that Λ_1' is the heavy brick and $(\Lambda_2' + \Lambda_3')$ is the square tetrad

$$[0'\ A'\ A']_a = \begin{array}{c} \begin{array}{ccc} \Lambda_1' & \Lambda_2' & \Lambda_3' \end{array} \\ \begin{array}{|c|c|c|} \hline x & x & x \\ \hline \begin{smallmatrix} x & \\ x & x \end{smallmatrix} & x & x \\ \hline \end{array} \end{array}.$$

Example III.2.11. As you see in Λ_2' is $E' + b$ and $|\Lambda_2'| = 4$, whereas $|\Lambda_1' + \Lambda_3'| = 4$. This implies that Λ_2' is the heavy brick and $(\Lambda_1' + \Lambda_3')$ is the square tetrad

$$[D\ E'\ A]_b = \begin{array}{c} \begin{array}{ccc} \Lambda_1' & \Lambda_2' & \Lambda_3' \end{array} \\ \begin{array}{|c|c|c|} \hline x & \begin{smallmatrix} x & x \\ x & \end{smallmatrix} & x \\ \hline x & & x \\ \hline \end{array} \end{array}.$$

Remark III.2.12. 1. There are 70 possibilities to arrange four points in eight places (so there are 70 heavy bricks). Suppose that x_1, x_2, x_3 and x_4 our four points in the heavy brick. Then, there are

1	2
3	4
5	6
7	8

$x_1 \mapsto 8$ Choices to put x_1 in any of the eight squares.
$x_2 \mapsto 7$ Choices to put x_2 in any of the eight squares.
$x_3 \mapsto 6$ Choices to put x_3 in any of the eight squares.
$x_4 \mapsto 5$ Choices to put x_4 in any of the eight squares.

Figure III.2.3: Arrange four points in eight places

However, $x_1 = x_2 = x_3 = x_4$, hence, the ordering is not important and there are repeated brick so to avoid this in figure III.2.3 we need to divide by $4!$. Therefore, $(8 \times 7 \times 6 \times 5)/4! = 70$ heavy bricks.

2. All possibilities to arrange four points in 16 places (square tetrad) is 140 bricks, assuming the property that the number of points in each columns should be equal to $(\mathrm{mod}\, 2)$, i.e. $2k \cong 0$, where $k \in \mathbb{Z}$. Similarly, for rows.

Shapes of columns There are $4 + (6 \times 6) + (6 \times 6) + 24 + (6 \times 6) + 4 = 140$ brick tetrad, see figure III.2.4.

	1	2	3	4
a	1	2	9	10
b	3	4	11	12
c	5	6	13	14
d	7	8	15	16

$(4\ 0\ 0\ 0) \times 4$, with $(1\,1\,1\,1) \times 1$ one point in each row.
$(2\ 2\ 0\ 0) \times 6$, with $(2\,2\,0\,0) \times 6$ shapes of rows.
$(2\ 2\ 0\ 0) \times 6$, with $(1\,1\,1\,1) \times 6$ one point in each row.
$(1\ 1\ 1\ 1) \times 24$, with $(1\ 1\ 1\ 1) \times 1$ one point in each row.
$(1\ 1\ 1\ 1) \times 6$, with $(2\ 2\ 0\ 0) \times 6$ shapes of rows.
$(1\ 1\ 1\ 1) \times 1$, with $(4\ 0\ 0\ 0) \times 4$ shapes of rows.

Figure III.2.4: Shapes of columns

Example III.2.13. Suppose that the shape of columns is $(4\ 0\ 0\ 0)$ and the shape of rows is $(1\ 1\ 1\ 1)$, all possibilities to arrange these shapes in 16 places are in figure III.2.5.

$$
\begin{array}{c|c|c|c|c|}
 & 4 & 0 & 0 & 0 \\ \hline
1 & x & & & \\ \hline
1 & x & & & \\ \hline
1 & x & & & \\ \hline
1 & x & & & \\ \hline
\end{array}\ ,\quad
\begin{array}{c|c|c|c|c|}
 & 0 & 4 & 0 & 0 \\ \hline
1 & & x & & \\ \hline
1 & & x & & \\ \hline
1 & & x & & \\ \hline
1 & & x & & \\ \hline
\end{array}\ ,
$$
$$
\begin{array}{c|c|c|c|c|}
 & 0 & 0 & 4 & 0 \\ \hline
1 & & & x & \\ \hline
1 & & & x & \\ \hline
1 & & & x & \\ \hline
1 & & & x & \\ \hline
\end{array}\ ,\quad
\begin{array}{c|c|c|c|c|}
 & 0 & 0 & 0 & 4 \\ \hline
1 & & & & x \\ \hline
1 & & & & x \\ \hline
1 & & & & x \\ \hline
1 & & & & x \\ \hline
\end{array}\ .
$$

Figure III.2.5: The shape of columns 4 0 0 0 and rows 1 1 1 1

Definition III.2.14. A picture contains a group of heavy bricks and a group of square bricks. Moreover, the 35 pictures are obtained from the one-to-one correspondence that is from 70 heavy bricks that are divided into two groups, (saying $|X| = |Y| = 4$ in the same group if

$X + Y = \Lambda$, see figure III.2.7), to 140 square tetrads that are divided into four groups, (saying $|U| = |V| = |W| = |Q| = 4$ in the same group if $U + V + W + Q = \Lambda + \Lambda$, see figure III.2.7).

Example III.2.15. Figure III.2.6 is the picture (1) in **MOG**, see figure III.3.1. This picture consists of a correspondent group of heavy bricks to a group of square bricks.

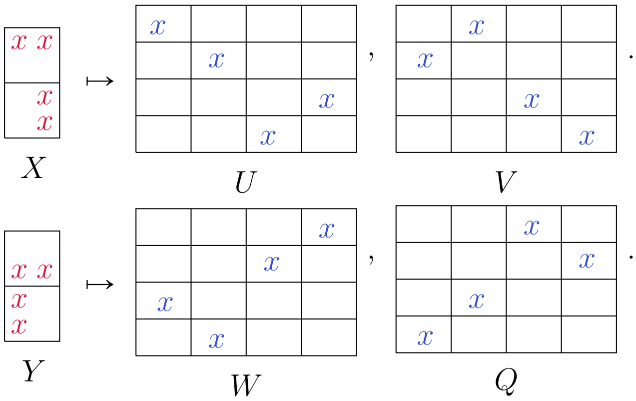


Figure III.2.6: Picture (1)

where

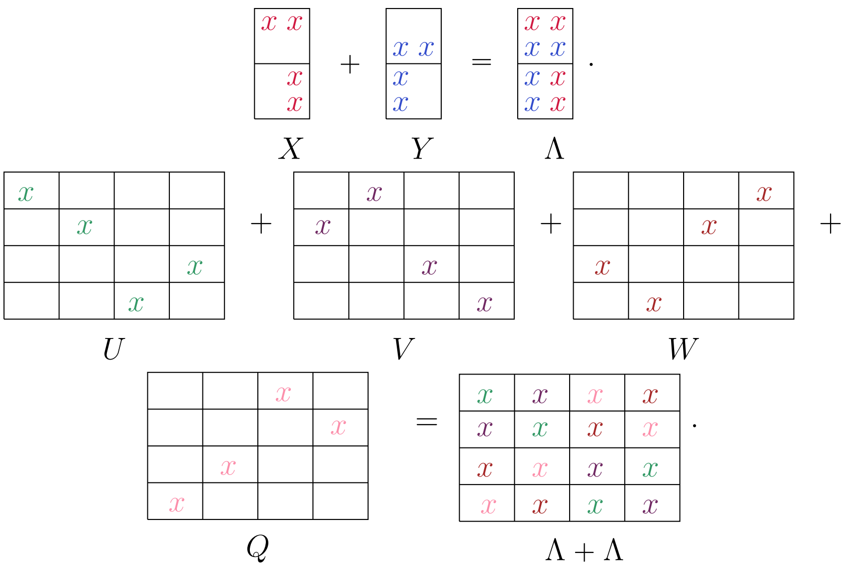


Figure III.2.7: The picture

III.3 The Miracle Octad Generator (MOG)

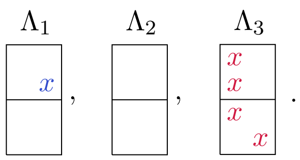
Definition III.3.1. The *Miracle Octad Generator* (**MOG**) is 36 pictures which are one of them shows named of points, whereas 35 pictures contain pair of brick tetrads and the corresponding group of square bricks. Moreover, taking any one of the pair together with any one of square bricks in the same group is an octad.

Remark III.3.2.

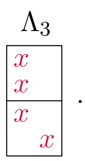
1. There are all the symmetries bodily permuting of $\Lambda_1, \Lambda_2, \Lambda_3$ in **MOG** figure, see figure III.3.1.
2. The heavy brick in these picture is in Λ_1. The square tetrad is in $\Lambda_2 + \Lambda_3$.
3. We obtain an octad by taking either of the brick tetrads together with any one of the square tetrads from the same picture.
4. red and blue present two different heavy bricks, and in the same picture, red square, blue square, purple circle and green circle present four different square tetrads.

Example III.3.3. [Cur] To find the octad that contains points $22, 1, 12, 6, 8$ in **MOG**, we need to do

Step 1: We should assign the points in $\Lambda_1, \Lambda_2, \Lambda_3$ by using picture (7) in **MOG**, see figure III.3.1



Step 2: Finding the heavy brick which here is Λ_3 (since $|\Lambda_3| = 4$) and it is in picture (31) in **MOG**, see figure III.3.1



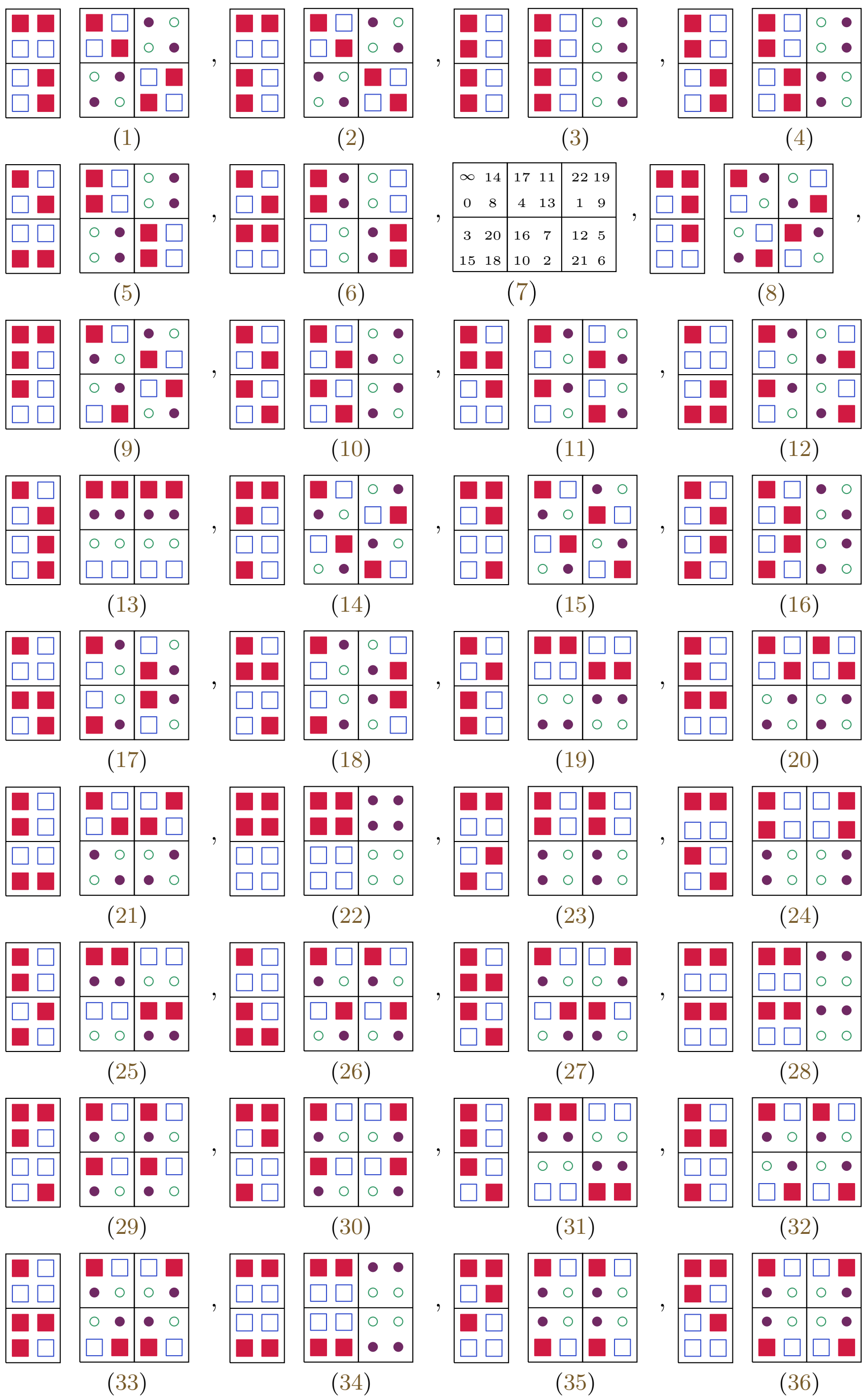
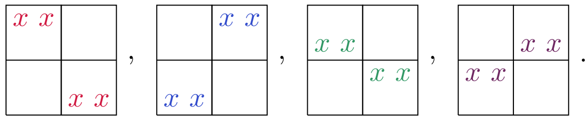


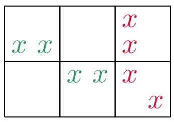
Figure III.3.1: The Miracle Octad Generator (MOG)

Step 3: Looking for square tetrad that contains (point 8), but in picture (31) there are four square tetrads



(I) (II) (III) (IV)

Hence, 8 is in Λ_1 this implies that (III) is the square tetrad which contains $(22, 1, 12, 6, 8)$ and then the octad is

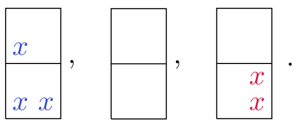


=(0,8,16,7,22,1,12,6).

Example III.3.4. [Cur] To find the octad that contains points 0, 15, 18,5, 6 in **MOG**, we need to do

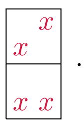
Step 1: We should assign the points in $\Lambda_1, \Lambda_2, \Lambda_3$ by using picture (7) in **MOG**, see figure III.3.1

Λ_1 Λ_2 Λ_3



Step 2: Finding the heavy brick which is Λ_1 (since $|\Lambda_1| = 3$) and it is in one of these pictures (6), (8), (28) or (35) in **MOG**, see figure III.3.1. However, Λ_3 should contain (points 5 and 6). Therefore, the heavy brick should be

Λ_3



Step 3: Looking for square tetrad that contains (points 5 and 6), but in picture (6) there are four square tetrads

$$
\underset{\text{(I)}}{\begin{array}{|cc|cc|}\hline x & & & \\ x & & & \\ \hline & & & x \\ & & & x \\ \hline\end{array}},\quad
\underset{\text{(II)}}{\begin{array}{|cc|cc|}\hline & & & x \\ & & & x \\ \hline x & & & \\ x & & & \\ \hline\end{array}},\quad
\underset{\text{(III)}}{\begin{array}{|cc|cc|}\hline & x & & \\ & x & & \\ \hline & & x & \\ & & x & \\ \hline\end{array}},\quad
\underset{\text{(IV)}}{\begin{array}{|cc|cc|}\hline & & x & \\ & & x & \\ \hline & x & & \\ & x & & \\ \hline\end{array}}.
$$

Since (points 5 and 6) are in Λ_3 this implies that (I) is the square tetrad which contains $(5, 6, 12, 4, 17)$ and then the octad is

$$
\begin{array}{|cc|cc|cc|}\hline & x & x & & & \\ x & & x & & & \\ \hline & & & & & x \\ x & x & & & & x \\ \hline\end{array} = (0,15,14,18,17,4,5,6).
$$

Example III.3.5. [Cur] To find the octad that contains points $0, 14, 2, 22, 21$ in **MOG**, we need to do

Step 1: We should assign the points in $\Lambda_1, \Lambda_2, \Lambda_3$ by using picture (7) in **MOG**, see figure III.3.1

$$
\overset{\Lambda_1}{\begin{array}{|cc|}\hline & x \\ x & \\ \hline & \\ & \\ \hline\end{array}},\quad
\overset{\Lambda_2}{\begin{array}{|cc|}\hline & \\ & \\ \hline & \\ & \\ \hline\end{array}},\quad
\overset{\Lambda_3}{\begin{array}{|cc|}\hline x & \\ x & \\ \hline x & \\ & x \\ \hline\end{array}}.
$$

Step 2: Finding the heavy brick which is either Λ_1 or Λ_3, since $|\Lambda_1| = 2 = |\Lambda_3|$. If Λ_3 is the heavy brick, then it is in one of these pictures (2, 3, 5, 12, 14, 16, 19, 21, 23, 25, 26, 30, 33 and 34). However, all the corresponding square tetrads in these pictures do not contain (points 14, 0 and 2) in any of their square tetrads. Therefore, Λ_1 must be the heavy brick which is in pictures (5, 6, 9, 10, 12, 13, 14, 16, 17, 19, 22, 26, 29, 30 and 36). Since square tetrad must contain (points 2, 22 and 21). This implies that the picture must be (36). Hence, the heavy brick is

$$
\overset{\Lambda_1}{\begin{array}{|cc|}\hline & x \\ x & \\ \hline & \\ & \\ \hline\end{array}}.
$$

Notice that, the square tetrad is

x	x
x	x

.

Hence, the octad is

x x x	x	x
x	x	x

$= (\infty, 0, 14, 20, 11, 22, 2, 21)$.

Corollary III.3.6. *Steiner systems $S(4,7,23)$ and $S(3,6,22)$ exist.*

Proof. Indeed, we can obtain $S(4,7,23)$ from Theorem III.1.2 and existence of $S(5,8,24)$. Moreover, the number of blocks from Theorem III.1.3 is $\binom{23}{4}/\binom{7}{4} = 253$ blocks. Similarly, $S(3,6,22)$ can be obtained from Corollary III.1.4 and existence of $S(5,8,24)$. The number of blocks from Theorem III.1.3 is $\binom{22}{3}/\binom{6}{3} = 77$ blocks. In general, let $S_8 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ be an octad and $S_j = \{a_1, a_2, a_3, \ldots, a_j\}$, where $j \leqslant 7$. Figure III.3.2 shows the number of octads intersecting S_i in S_j, where $(j+1)$ is the entry and $(i+1)$ is the line. As an example, let $\alpha, \beta \in \Omega$ then there are $253 - 77 = 176$ octads that contain α not β. $\square$

$$
\begin{array}{c}
759 \\
506 \quad 253 \\
330 \quad 176 \quad 77 \\
220 \quad 120 \quad 56 \quad 21 \\
130 \quad 80 \quad 40 \quad 16 \quad 5 \\
78 \quad 52 \quad 28 \quad 12 \quad 4 \quad 1 \\
46 \quad 32 \quad 20 \quad 8 \quad 4 \quad 0 \quad 1 \\
30 \quad 16 \quad 16 \quad 4 \quad 4 \quad 0 \quad 0 \quad 1 \\
30 \quad 0 \quad 16 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0 \quad 1
\end{array}
$$

Figure III.3.2: The Leech triangle

Remark III.3.7. 1. The ninth line in figure III.3.2 shows that any two octads intersect in 0, 2, 4 or 8 points.

2. There is another way to count how many octads contain i points which is $\binom{24-i}{5-i}/\binom{8-i}{5-i}$, where $0 \leqslant i \leqslant 4$. Moreover, it is 1, where $i \geqslant 5$.

Lemma III.3.8. *[Cur, Lemma 1] If $S, T \in \mathscr{C}_8$ and $|S \bigcap T| = 4$, then $S + T \in \mathscr{C}_8$.*

Proof. Let $S = \{a_1, a_2, \ldots, a_8\}$, $T = \{a_1, a_2, a_3, a_4, b_5, b_6, b_7, b_8\}$ be two octads and suppose $T+S \notin \mathscr{C}_8$. Considering another octad, let say W which contains $(a_5, a_6, a_7, a_8, b_5)$. From Remark III.3.7 there are no two octads which intersect in one point. Therefore, W contains a further point of T and not a's since then $|S \bigcap W| \geqslant 5$, so, let say b_6. Similarly, with W_1 that contains $(a_5, a_6, a_7, a_8, b_7)$, let say b_8. However, considering the octad W_2 that contains $(a_5, a_6, a_7, b_5, b_7)$ implies that W_2 must contain a further point of S. If $a_8 \in W_2$ then $|W_1 \bigcap W_2| \geqslant 5$, let say a_1, but then W_2 must contain another point of T. If a's is added then $|S \bigcap W_2| \geqslant 5$, and if b_8 is added then $|W_1 \bigcap W_2| \geqslant 5$. Moreover, if b_6 is added then $|T \bigcap W_2| \geqslant 5$. In each case we reach to a contradiction, hence $T + S \in \mathscr{C}_8$.

□

Definition III.3.9. Let $Y = Y_1 \cup Y_2 \cdots \cup Y_s$ be a decomposition of Y into disjoint sets Y_i, and X is a subset of Y. If $|Y_i \bigcap X| = r_i$ points, $1 \leqslant i \leqslant s$ then X cuts this decomposition as $r_1.r_2.\cdots.r_s$, where $|X| = r_1 + r_2 + \cdots + r_s$.

Corollary III.3.10. *[Cur] There is a partition of the twenty-four points into six tetrads, which is an correspondence to each four-point of Ω, let say Y_i, $i = 1, \ldots, 6$ then $\Omega = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6$, where $|Y_i| = 4, |Y_i + Y_j| = 8$, $i \neq j$ and $i, j = 1, \ldots, 6$. Moreover, $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ is called a sextet.*

Lemma III.3.11. *[Cur, Lemma 2] An octad cuts the six tetrads of a sextet $4^2 \cdot 0^4$, $3 \cdot 1^5$ or $2^4 \cdot 0^2$.*

Proof. Let Y be a set of 24 points, $X \subseteq Y, |X| = 8$, and $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6$. There are three cases,

Case 1. If $|Y_i \bigcap X| = 1$ point, where $i = 1, \ldots, 6$ and $|X| = 8$, then there are two points left, if the two points in the same Y_i, let say Y_1 then $|Y_1 \bigcap X| = 3$. Since $|(Y_i + Y_j) \bigcap X| = 2$ or 4, where $i \neq j$ and $i, j = 1, \ldots, 6$. Hence, X cuts Y_i as 3.1.1.1.1.1, whereas, if the two points in different Y_i, let say Y_1 and Y_2 then $|(Y_1 + Y_3) \bigcap X| = 3$ which is a contradiction with Lemma III.3.11.

Case 2. If $|Y_i \bigcap X| = 2$ points, where $i = 1, \ldots, 6$ and $|X| = 8$ then there are only four Y_i, let say Y_1, Y_2, Y_3 and Y_4 which intersect with X in two points. Hence, $|(Y_i + Y_j) \bigcap X| = 0, 2$ or 4, where $i \neq j$ and $i, j = 1, \ldots, 6$. Hence, X cuts Y_i as 2.2.2.2.0.0.

Case 3. If $|Y_i \bigcap X| = 4$ points, where $i = 1, \ldots, 6$ and $|X| = 8$ then there are only two Y_i, let say Y_1 and Y_2 which intersect with X in four points. Hence, $|(Y_i + Y_j) \bigcap X| = 0, 4$ or 8, where $i \neq j$ and $i, j = 1, \ldots, 6$. Hence, X cuts Y_i as 4.4.0.0.0.0.

$\square$

Lemma III.3.12. *[Cur, Lemma 3] The intersection matrix for the tetrads of two sextets is one of the following:*

$$
\text{(I)} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \text{(II)} \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix},
$$

$$
\text{(III)} \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \text{(IV)} \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.
$$

Proof. Let $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ and $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$ be the two sextets. Suppose that $Y_i + Y_j$ is an octads and Z_k is a sextet, where $i \neq j$, $i, j = 1, \ldots, 6$ and $k = 1, \ldots, 6$. By using Lemma III.3.11 we get these matrices, where the entry in ith row and j^{th} column is the intersection of Y_i and Z_j.

$\square$

Example III.3.13.

$$\begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{array} \begin{array}{c} \begin{array}{cccccc} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 \end{array} \\ \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Let $Y_1 + Y_2$ be an octad then it cuts $Z_1 \dot\cup Z_2 \dot\cup Z_3 \dot\cup Z_4 \dot\cup Z_5 \dot\cup Z_6$ as 4.4.0.0.0.0. Suppose that $Y_1 + Y_6$ is an octad then it cuts $Z_1 \dot\cup Z_2 \dot\cup Z_3 \dot\cup Z_4 \dot\cup Z_5 \dot\cup Z_6$ as 3.1.1.1.1.1.

Theorem III.3.14. *[Cur, Theorem B] The Steiner system $S(5, 8, 24)$ is unique.*

Proof. Suppose that $\Omega = \{\infty, 0, 1, \ldots, 22\}$ and $O_1 \subseteq \Omega$ is an octad such that x_1, x_2, x_3, x_4, x_5, x_6 in O_1 and $x_7 \in \Omega \backslash O_1$. Notice that, we can write $O_1 + \Omega$ in 4×6 array where the first two columns are O_1

$$\begin{array}{|cc|cc|cc|} \hline x_1 & x_5 & x_7 & & & \\ x_2 & x_6 & & & & \\ \hline x_3 & & & & & \\ x_4 & & & & & \\ \hline \end{array}.$$

Assuming that S_∞ is a sextet such that it defines by the tetrad $Y_x = \{x_1, x_2, x_3, x_4\}$. Then by rearranging the un-named 17 points we get this

$$S_\infty = \begin{array}{|cc|cc|cc|} \hline x & 0 & 1 & 2 & 3 & 4 \\ x & 0 & 1 & 2 & 3 & 4 \\ \hline x & 0 & 1 & 2 & 3 & 4 \\ x & 0 & 1 & 2 & 3 & 4 \\ \hline \end{array}.$$

Notice that, $S_\infty = Y_x \cup Y_0 \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4$, which means that S_∞ is a sextet. Moreover, O_1 cuts S_∞ as 4.4.0.0.0.0. Notice that, considering the octad, let say O_2 such that $x_2, x_3, x_4, x_5, x_6, x_7 \in O_2$. Then O_2 cuts S_∞ as 3.1.1.1.1.1 and

$$S_0 = \begin{array}{|cc|cc|cc|} \hline 0 & x & 1 & 1 & 1 & 1 \\ x & 0 & 2 & 2 & 2 & 2 \\ \hline x & 0 & 3 & 3 & 3 & 3 \\ x & 0 & 4 & 4 & 4 & 4 \\ \hline \end{array} \text{ is a sextet.}$$

Notice that, from figure III.3.2 the number of disjoint octads from O_1 is 30 octads. If we consider the octad, let say O_3 such that $x_1, x_3, x_4, x_5, x_7 \in O_3$. Then O_3 cuts both S_∞ and S_0 as 3.1.1.1.1.1, where $|O_3 \bigcap Y_x| = 3$ points and $|O_3 \bigcap Y_1| = |O_3 \bigcap Y_2| = |O_3 \bigcap Y_3| = |O_3 \bigcap Y_4| = 1$ point. This implies that

$$S_1 = \begin{array}{|cc|cc|cc|} \hline x & x & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 1 & 4 & 3 \\ \hline x & 0 & 3 & 4 & 1 & 2 \\ x & 0 & 4 & 3 & 2 & 1 \\ \hline \end{array} \text{ is a sextet.}$$

Notice that, the group of permutations that preserve the sextets S_∞, S_0 and S_1 is given by

$$\pi = \begin{array}{|cc|cc|cc|} \hline \bullet & \bullet & \bullet & a_1 & a_2 & a_3 \\ \bullet & \bullet & c_1 & b_1 & d_2 & e_3 \\ \hline \bullet & \bullet & c_2 & e_1 & b_2 & d_3 \\ \bullet & \bullet & c_3 & d_1 & e_2 & b_3 \\ \hline \end{array}, \quad \alpha = \begin{array}{|cc|cc|cc|} \hline \bullet & \bullet & \bullet & \bullet & \multicolumn{2}{c|}{—} \\ \bullet & \bullet & \bullet & \bullet & \multicolumn{2}{c|}{—} \\ \hline \bullet & \bullet & | & | & \multicolumn{2}{c|}{\times} \\ \bullet & \bullet & | & | & \multicolumn{2}{c|}{} \\ \hline \end{array}, \quad \rho = \begin{array}{|cc|cc|cc|} \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & | & \bullet & \bullet & \bullet & \bullet \\ \bullet & | & \bullet & \bullet & \bullet & \bullet \\ \hline \end{array},$$

where dots denote fixed points and π is a 3-element taking

$$x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_1,$$

where $x_i \in \{a_i, b_i, c_i, d_i, e_i\}$. Let O_4 be an octad such that $x_1, x_2, x_5, x_6, x_7 \in O_4$. Then O_4 cuts S_∞ as 2.2.2.2.0.0. Let assume that O_4 cuts the first four columns of S_∞ by using π it is as 2.2.2.2. Also, O_4 cuts S_0 as 2.2.2.2.0.0. Thus, we got the top two points of the fourth columns

$$\begin{array}{|cc|cc|cc|} \hline x & x & x & x & & \\ x & x & x & x & & \\ \hline & & & & & \\ & & & & & \\ \hline \end{array}.$$

Thus,

$$S_2 = \begin{array}{|cc|cc|cc|} \hline x & x & 1 & 1 & 2 & 2 \\ x & x & 1 & 1 & 2 & 2 \\ \hline 0 & 0 & 3 & 3 & 4 & 4 \\ 0 & 0 & 3 & 3 & 4 & 4 \\ \hline \end{array} \text{ is a sextet.}$$

Notice that, $O_4 = Y_x \cup Y_1$. Let O_5 contain x_1, x_2, x_3, x_5, x_7 such that $O_5 \neq O_4, O_3, O_2$ or O_1 and O_5 cuts S_∞, S_0 and S_1 as 3.1.1.1.1.1. Therefore, O_5 must be

$$\begin{array}{|cc|cc|cc|} \hline x & x & x & & & \\ x & & & & & x \\ \hline x & & & x & & \\ & & & & x & \\ \hline \end{array} \text{ or } \begin{array}{|cc|cc|cc|} \hline x & x & x & & & \\ x & & & & x & \\ \hline x & & & & & x \\ & & & x & & \\ \hline \end{array}.$$

Figure III.3.3: An octad

By using the permutation α these are equivalent. Let assume that

$$S_3 = \begin{array}{|cc|cc|cc|} \hline x & x & 1 & 2 & 3 & 4 \\ x & 0 & 3 & 4 & 1 & 2 \\ \hline x & 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 & 4 & 3 \\ \hline \end{array} \text{ is a sextet.}$$

Notice that, from figure III.3.3

$$S_4 = \begin{array}{|cc|cc|cc|} \hline x & x & 1 & 2 & 3 & 4 \\ x & 0 & 4 & 3 & 2 & 1 \\ \hline 0 & 0 & 2 & 1 & 4 & 3 \\ x & 0 & 3 & 4 & 1 & 2 \\ \hline \end{array} \text{ is a sextet.}$$

Let consider the octad that contains these points

$$\begin{array}{|cc|cc|cc|} \hline x & x & x & & & \\ & & & & & \\ \hline x & & x & & & \\ & & & & & \\ \hline \end{array}.$$

It must have further points in the second column and two points in one of the last three columns, but since it cuts S_∞, S_0 and S_1 as 2.2.2.2.0.0, we get

$$\begin{array}{|cc|cc|cc|}\hline x & x & x & x & & \\ & & & & & \\ \hline x & & x & x & & \\ & & & & & \\ \hline \end{array}.$$

For the one point further that is in the second column we might have one of the following

$$\underset{\text{(I)}}{\begin{array}{|cc|cc|cc|}\hline x & x & x & x & & \\ & x & & & & \\ \hline & x & x & x & & \\ & & & & & \\ \hline \end{array}}, \quad \underset{\text{(II)}}{\begin{array}{|cc|cc|cc|}\hline x & x & x & x & & \\ & & & & & \\ \hline x & x & x & x & & \\ & & & & & \\ \hline \end{array}}, \quad \underset{\text{(III)}}{\begin{array}{|cc|cc|cc|}\hline x & x & x & x & & \\ & & & & & \\ \hline x & & x & x & & \\ & x & & & & \\ \hline \end{array}}.$$

The first case (I) fails to cut S_2. (Because it cuts as 3.1.2.2.0.0 which is a contradiction with Lemma III.3.11, wheras (II) and (III) are equivalent under ρ, so we might take (II) as our octad

$$\begin{array}{|cc|cc|cc|}\hline x & x & x & x & & \\ & & & & & \\ \hline x & x & x & x & & \\ & & & & & \\ \hline \end{array}.$$

Thus,

$$S_5 = \begin{array}{|cc|cc|cc|}\hline x & x & 1 & 1 & 3 & 3 \\ 0 & 0 & 2 & 2 & 4 & 4 \\ \hline x & x & 1 & 1 & 3 & 3 \\ 0 & 0 & 2 & 2 & 4 & 4 \\ \hline \end{array} \text{ is a sextet.}$$

Now, we get S_∞, S_0, S_1,S_2, S_3, S_4, S_5 and to obtain the 28 sextets remaining, we need the following Lemma. □

Lemma III.3.15. *[Cur, Lemma 4] If every octad intersecting a given octad O in four points is known, then all octads follow by symmetric differencing.*

Proof. Let O be the given octad and $x, y, z \in O$, which are distinct points. From figure III.3.2 there are 21 octads containing x, y, z. However, from Remark III.3.7 and Lemma III.3.8 any two octads intersect in 0, 2, 4 or 8 points. Therefore, the intersection must be in four points so their symmetric difference is an octad disjoint from x, y, z. Notice that, there are $\binom{21}{2} = 210$ pairs, which are disjoint from x, y, z in 21 octads. Since suppose that U_i is an octad such that $x, y, z \in U_i$, where $i = 1, 2, 3, 4$. If $U_1 + U_2 = U_3 + U_4$ then U_3 or U_4 must contain further two points from $U_1 \backslash \{x, y, z\}$, let say U_3. Thus, $|U_1 \bigcap U_3| = 5$, $U_1 = U_3$ and this implies that $U_2 = U_4$. Therefore, all 210 pairs are unique.

Using figure III.3.2 again the third line consists of all the disjoint octads from x, y, z. Hence, we know every octad that is disjoint from O by three points. $\square$

Corollary III.3.16. *The set of all permutations of* 24*-element,* Ω *that preserve* $S(5, 8, 24)$*, and it has form quintuply transitive is a subgroup of symmetric group of* 24*-element,* S_{24}*. Moreover it has order 244, 823, 040.*

Proof. Notice that, the set of all the permutations is a group and it is sharply transitive on sets which contain 7 points, let say $x_1, x_2, x_3, x_4, x_5, x_6, x_7$, where $x_1, x2, x_3, x_4, x_5, x_6$ in the same octad O, whereas $x_7 \in \Omega \backslash \{0\}$. There are 24 ways of choosing x_1 from Ω, 23 choices for choosing x_2 and there are $22, 21, 20$ choices for x_3, x_4, x_5, respectively. Notice that, $x_6 \in O$ and $|O| = 8$. Hence, there are three choices for x_6. Finally, there are 16 choices for choosing $x_7 \in \Omega \backslash O$. Therefore, $24 \times 23 \times 22 \times 21 \times 20 \times 3 \times 16 = 244, 823, 040$.

$\square$

Definition III.3.17. The 5-transitive group preserving $\mathscr{C}_8$ is called $M_{24} = \{\sigma \in S_{24} \mid O_\sigma \in \mathscr{C}_8, \text{ for all } O \in \mathscr{C}_8\}$. Moreover, subgroups of M_{24} are M_{24k}, where $(k < 5)$ which are fixed k-points.

Remark III.3.18. The points of Ω are $\infty, 0, 1, 2, \ldots, 22$ and they number as the project line. Suppose that $\sigma \in M_{24}$ and $o(\sigma) = 23$, (notice that $|M_{24}|/23$) such that $\sigma : i \mapsto i + 1$ (mod 23) and it fixes ∞, i.e. $\sigma = (\infty)(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22)$. Let $\gamma \in \Omega$ such that $\gamma : i \mapsto \frac{-1}{i}$. This implies that $\gamma = (0\ \infty)(1\ 22)\ (2\ 11)(3\ 15)\ (4\ 17)\ (5\ 9)\ (6\ 19)(7\ 13)\ (8\ 20)(10\ 16)\ (12\ 21)(14\ 18)$. Thus, $\gamma \in M_{24}$ since $(S_i)\gamma$, where $i = \infty, 0, 1, 2, 3, 4, 5$ is a sextet.

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