# Poisson algebras and iterated skew polynomial algebras

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A nonempty set $S'$ with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

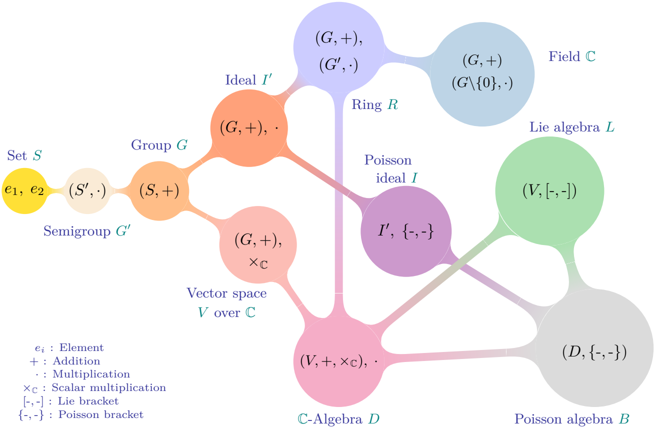
1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set $S$ with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. $S$ is an abelian if $g + h = h + g$.

A nonempty set $V$ with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field $\mathbb{C}$ if for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
6. $1 \times v = v$.




$e_i$ : Element
$+$ : Addition
$\cdot$ : Multiplication
$\times_{\mathbb{C}}$ : Scalar multiplication
$[\text{-},\text{-}]$ : Lie bracket
$\{\text{-},\text{-}\}$ : Poisson bracket

Figure 1: Algebraic structure

# Outline

# Poisson algebras

**Definition 1**

A (commutative) $\mathbb{C}$-algebra $(D, +, \cdot)$ is said to be a *Poisson algebra* if there exists bilinear product $\{\text{-},\text{-}\}$ on $D$, called a Poisson bracket, such that $(D, \{\text{-},\text{-}\})$ is

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$ for all $a, b, c \in D$ (Leibniz rule).

# Skew polynomial rings

**Definition 2**

Let $\mathbb{C}$ be a ring and $\beta$ be an automorphism of $\mathbb{C}$ and $\sigma$ an $\beta$-derivation on $\mathbb{C}$. Then $A = \mathbb{C}[x; \beta, \sigma]$ is called a *skew polynomial ring over* $\mathbb{C}$ if

1. $A$ is a ring, containing $\mathbb{C}$ as a subring,
2. $x$ is an element of $A$,
3. $A$ is a free left $\mathbb{C}$-module with basis $\{1, x, x^2, \ldots\}$, and
4. $xr = \beta(r)x + \sigma(r)$ for all $r \in \mathbb{C}$.

# Semiclassical limit algebras

**Definition 3**

A nonzero element $t-1$ in a Poisson algebra $D$ is called a *regular element* of $D$, if $t-1$ is a nonunit, nonzero divisor and central element of $D$ such that $D/(t-1)D$ is commutative.

**Definition 4**

Let $D$ be a $\mathbb{C}$-algebra and $t-1 \in D$ be a regular element. Then a nontrivial commutative algebra $D/(t-1)D$ is a Poisson algebra with Poisson bracket defined by the rule

$$\{\bar{a}, \bar{b}\} = \overline{(t-1)^{-1}(ab-ba)} \tag{1}$$

for $\bar{a}, \bar{b} \in D/(t-1)D$. The Poisson algebra $D/(t-1)D$ is called a *semiclassical limit* of $D$.

# The construction

**Theorem 5** [MyOh]

*Let* $\mathbb{F}$ *subring of* $\mathbb{C}[[t-1]]$ *and* $A_{k-1} = \mathbb{F}[x_1][x_2;\beta_2,\sigma_2]\ldots[x_{k-1};\beta_{k-1},\sigma_{k-1}]$ *be an iterated skew polynomial* $\mathbb{F}$*-algebra and assume that* $\beta_j, \sigma_j, a_{ji}, u_{ji}$ *satisfy*

$$\beta_j(x_i) = a_{ji}x_i, \quad a_{ji} \in \mathbb{F} \qquad (1 \le i < j \le k-1), \tag{2}$$

$$\sigma_j(x_i) = u_{ji} \in A_i \qquad (1 \le i < j \le k-1). \tag{3}$$

*Let* $\beta_k$, $\sigma_k$ *be* $\mathbb{F}$*-linear maps, from* $A_{k-1}$ *into itself subject to*

$$\beta_k(1) = 1, \quad \beta_k(x_i) = a_{ki}x_i, \;\; a_{ki} \in \mathbb{F} \qquad (1 \le i \le k-1), \tag{4}$$

$$\sigma_k(1) = 0, \quad \sigma_k(x_i) = u_{ki} \in A_i \qquad (1 \le i \le k-1), \tag{5}$$

*if* $\beta_k$ *and* $\sigma_k$ *satisfy the conditions*

$$\beta_k(u_{ji}) = a_{kj}a_{ki}u_{ji} \qquad (1 \le i < j < k), \tag{6}$$

$$a_{kj}x_ju_{ki} + u_{kj}x_i = a_{ji}u_{ki}x_j + a_{ki}a_{ji}x_iu_{kj} + \sigma_k(u_{ji}) \qquad (1 \le i < j < k). \tag{7}$$

*Then there exists an iterated skew polynomial* $\mathbb{F}$*-algebra*

$$A_k = A_{k-1}[x_k;\beta_k,\sigma_k] = \mathbb{F}[x_1][x_2;\beta_2,\sigma_2]\ldots[x_k;\beta_k,\sigma_k].$$

## Theorem 6 [MyOh]

*Let* $A_k = \mathbb{F}[x_1][x_2;\beta_2,\sigma_2]\dots[x_k;\beta_k,\sigma_k]$ *be the iterated skew polynomial* $\mathbb{F}$*-algebra that is in Theorem 5. Suppose that* $\mathbb{F}/(t-1)\mathbb{F}$ *is isomorphic to* $\mathbb{C}$ *such that* $t-1$ *is a nonunit and nonzero divisor in* $A_k$ *and*

$$a_{ji} - 1 \in (t-1)\mathbb{F}, \quad \sigma_j(x_i) \in (t-1)A_k \tag{8}$$

*for all* $1 \le i < j \le k$*. Then* $t-1$ *is a regular element of* $A_k$ *and the semiclassical limit* $\overline{A}_k = A_k/(t-1)A_k$ *is Poisson isomorphic to an iterated Poisson polynomial* $\mathbb{C}$*-algebra*

$$\mathbb{C}[x_1][x_2;\alpha_2,\delta_2]_p\dots[x_k;\alpha_k,\delta_k]_p, \tag{9}$$

*where*

$$\alpha_j(x_i) = \left(\frac{da_{ji}}{dt}|_{t=1}\right)x_i, \quad \delta_j(x_i) = \frac{d\sigma_j(x_i)}{dt}|_{t=1} \tag{10}$$

*for all* $1 \le i < j \le k$.

**Lemma 7** [MyOh]

*Let $B_k = \mathbb{C}[x_1, \ldots, x_k]$ be a Poisson algebra satisfying the following condition: for any $1 \le i < j \le k$,*

$$\{x_j, x_i\} = c_{ji} x_i x_j + p_{ji}, \tag{11}$$

*where $c_{ji} \in \mathbb{C}$ and $p_{ji} \in B_i$. Then $B_k$ is an iterated Poisson polynomial algebra of the form*

$$B_k = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \ldots [x_k; \alpha_k, \delta_k]_p, \tag{12}$$

*where*

$$\alpha_j(x_i) = c_{ji} x_i, \quad \delta_j(x_i) = p_{ji}. \tag{13}$$

*Conversely, if $B_k$ is an iterated Poisson polynomial algebra of the form (12) then $B_k$ is a Poisson algebra satisfying the condition (11).*

## Corollary 8 [MyOh]

*Let $B_k$ be an iterated Poisson polynomial $\mathbb{C}$-algebra*

$$B_k = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \dots [x_k; \alpha_k, \delta_k]_p$$

*such that*

$$\alpha_j(x_i) = c_{ji}x_i, \qquad \delta_j(x_i) \in \mathbb{C}[x_1, \dots, x_i], \tag{14}$$

*where $c_{ji} \in \mathbb{C}$, for all $1 \leq i < j \leq k$ and let*

$$a_{ji} \in \mathbb{F}, \quad u_{ji} \in \mathbb{F}[x_1, \dots, x_i]$$

*such that*

$$\begin{aligned} a_{ji} - 1 &\in (t-1)\mathbb{F}, & \frac{da_{ji}}{dt}|_{t=1} &= c_{ji}, \\ u_{ji} &\in (t-1)\mathbb{F}[x_1, \dots, x_i], & \frac{du_{ji}}{dt}|_{t=1} &= [\delta_j(x_i)], \end{aligned} \tag{15}$$

*where $[\delta_j(x_i)]$ is the $\mathbb{C}$-linear combination of $\delta_j(x_i)$ by standard monomials of $x_1, \dots, x_i$.*

*Set $A_1 = \mathbb{F}[x_1]$. Suppose that $\mathbb{F}/(t-1)\mathbb{F}$ is isomorphic to $\mathbb{C}$ and that $t-1$ is a nonunit and nonzero divisor of an iterated skew polynomial $\mathbb{F}$-algebra*

$$A_{k-1} = \mathbb{F}[x_1][x_2; \beta_2, \sigma_2] \dots [x_{k-1}; \beta_{k-1}, \sigma_{k-1}]$$

*such that $\beta_j, \sigma_j$ satisfy some relations in [MyOh] for all $1 \leq i < j \leq k$. Then there exists an iterated skew polynomial $\mathbb{F}$-algebra*

$$A_k = A_{k-1}[x_k; \beta_k, \sigma_k] = \mathbb{F}[x_1][x_2; \beta_2, \sigma_2] \dots [x_{k-1}; \beta_{k-1}, \sigma_{k-1}][x_k; \beta_k, \sigma_k] \tag{16}$$

*and $t-1$ is a regular element of $A_k$ such that $B_k$ is Poisson isomorphic to the semiclassical limit $A_k/(t-1)A_k$.*

## Remark 9

Keep the notations of Theorem 5. If $u_{ji} = 0$ for all $1 \leq i < j \leq k$ then (6) and (7) hold trivially.

## Example 10

The commutative $\mathbb{C}$-algebra $B_3 = \mathbb{C}[x, y, z]$ is a Poisson $\mathbb{C}$-algebra with Poisson bracket defined by the rule

$$\{y, x\} = xy, \quad \{z, x\} = xz \quad \text{and} \quad \{z, y\} = yz. \tag{17}$$

Notice that, $B_3$ is an iterated Poisson polynomial $\mathbb{C}$-algebra

$$B_3 = \mathbb{C}[x][y; \alpha_2]_p[z; \alpha_3]_p, \tag{18}$$

where $\alpha_2(x) = x = \alpha_3(x)$ and $\alpha_3(y) = y$. Set $\mathbb{F} = \mathbb{C}[t]$ and $a_{21} = a_{31} = a_{32} = t$. Then, by using Remark 9 and Theorem 5, there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A_3 = \mathbb{F}[x][y; \beta_2][z; \beta_3], \tag{19}$$

where $\beta_2(x) = a_{21}x = \beta_3(x) = a_{31}x = tx$, and $\beta_3(y) = a_{32}y = ty$. Notice that, $t - 1$ is a regular element of $A_3$. By using Corollary 8 the semiclassical limit $A_3/(t-1)A_3$ is Poisson isomorphic to $B_3$ since (15) holds, i.e.

$$a_{ji} - 1 \in (t-1)\mathbb{F}, \quad \frac{da_{ji}}{dt}|_{t=1} = 1$$

for $1 \leq i < j \leq 3$

- References

[GoWa] K. R. Goodearl and R. B. Warfield. *An introduction to noncommutative noetherian rings*. 2nd ed. New York: Cambridge University Press. (2004), pages 1–85, 105–122 and 166–186.

[MyOh] N.-H. Myung and S.-Q. Oh, A construction of an Iterated Ore extension. arXiv:1707.05160v2 (2018).

[Oh2] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006) 1265–1277.

Thank you for listening

# Further resources

Poisson algebra applications :

- Skew polynomial rings.

[MyOh] N.-H. Myung and S.-Q. Oh, A construction of an Iterated Ore extension. arXiv:1707.05160v2 (2018).

- Poisson polynomial ring.

[Oh2] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265–1277.

- Poisson modules and Poisson enveloping algebras.

[Oh1] S.-Q. Oh, Poisson enveloping algebras. *Communications in Algebra*, **27** (1999), 2181–2186.

- Weyl algebras and generalised algebras.

[Bav] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

# Further resources

- Quantization and deformation algebras.

[BPR] C. Beem, W. Peelaers and L. Rastelli, Deformation quantization and superconformal symmetry in three dimensions. *Comm. Math. Phys.* **354** (2017), 345–392.

- Semiclassical limits.

[ChOh] E.-H. Cho and S.-Q. Oh, Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra. *Lett. Math. Phys.* **106** (2016), 997–1009.

- Star product and Poisson algebras.

[EtSt] P. Etingof and D. Stryker, Short star-products for filtered quantizations, I. arXiv:1909.13588v5 (2020).

- Poisson manifolds.

[Kon] M. Kontsevich, Deformation quantization of Poisson manifolds. arXiv:q-alg/9709040v1 (1997).

- Hamiltonian systems.