Poisson algebras
Non-commutative algebras

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A nonempty set S' with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set S with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. S is an abelian if $g + h = h + g$.

A nonempty set V with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field $\mathbb{C}$ if for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
6. $1 \times v = v$.

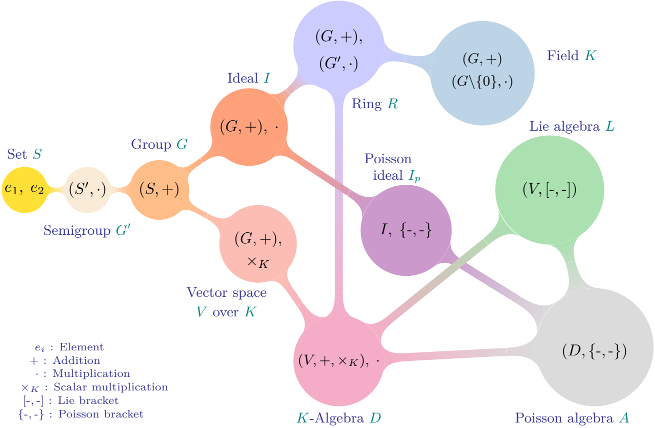

Figure 1: Algebraic structure

Outline

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Definition 1

A *ring* R is a set on which two binary operations are defined addition $(+)$ and multiplication $(\cdot)$ such that

1. $(R,+)$ is an abelian group,
2. $(R,\cdot)$ is semigroup with an identity element, and
3. The distributive laws hold for all $a,b,c \in R$

$$a \cdot (b+c) = a \cdot b + a \cdot c \text{ and } (a+b) \cdot c = a \cdot c + b \cdot c.$$

We say K is a field if K satisfies 1, 3 and $(K \backslash \{0\} = K^{\times}, \cdot)$ is an abelian group.

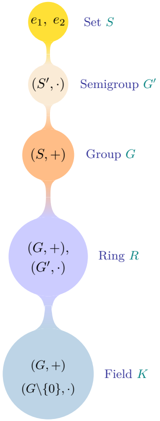

Figure 2: Rings and fields structures

Definition 2

Let K be a field and D be a set on which three operations are defined, addition, multiplication, and multiplication by a scalar such that

1. $(D, +, \cdot)$ is a ring,
2. $(D, +, \times_K)$ is a vector space over K,
3. $(\lambda a)\, b = a\, (\lambda b) = \lambda\, (ab)$ for all $\lambda \in K$ and $a, b \in D$,

then D is called a K-*algebra*

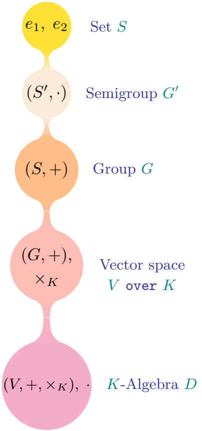

Figure 3: K-algebras structure

Definition 3

A (commutative) K-algebra $(D, +, \cdot)$ is called a *Poisson algebra* if there exists bilinear product $\{\text{-},\text{-}\}$ on D, called a Poisson bracket, such that $(D, \{\text{-},\text{-}\})$ is

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{a \cdot b, c\} = \{a, c\} \cdot b + a \cdot \{b, c\}$ for all $a, b, c \in D$ (Leibniz rule).

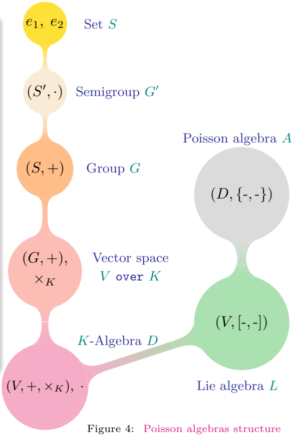

Figure 4: Poisson algebras structure

Definition 4

Let D be a Poisson algebra. A subset I of D is a *Poisson ideal* of D if

1. I is an ideal of algebra D, and
2. $\{d, a\} \in I$ for all $d \in D$ and $a \in I$.

Moverover, the algebra D is a *simple Poisson* algebra if the only Poisson ideals of D are D and 0.

Definition 5

Let D be a Poisson algebra. A Poisson ideal P is a *Poisson prime ideal* of D if the following satisfies:

$$IJ \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P$$

where I and J are Poisson ideals of D. Moreover, a set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

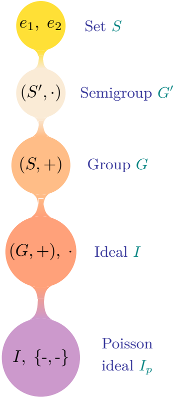

Figure 5: Poisson ideals structure

The Poisson centre, Derivations and Poisson derivations

Definition 6

Let D be a Poisson algebra then

$$\mathrm{PZ}(D) := \{a \in D \mid \{a, d\} = 0 \text{ for all } d \in D\}$$

is called the *Poisson centre* of D.

Definition 7

Let D be a associative Poisson algebra over K. A K-linear map $\alpha : D \to D$ is called a *derivation* (respectively, *Poisson derivation*) on D if α satisfies 1 (respectively, satisfies 1 and 2) of the following conditions:

1. $\alpha(a \cdot b) = \alpha(a) \cdot b + a \cdot \alpha(b)$ for all $a, b \in D$.
2. $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in D$.

A set of all *derivations* (respectively, *Poisson derivations*) on D denoted by $\mathrm{Der}_K(D)$ (respectively, $\mathrm{PDer}_K(D)$).

Theorem 8 [Oh2, Theorem 1.1]

Let D be a Poisson algebra over K and α, δ be K-linear maps on D. Then the polynomial ring $D[y]$ becomes a Poisson algebra with Poisson bracket:

$$\{a, y\} = \alpha(a)y + \delta(a) \quad \textit{for all } a \in D \tag{1}$$

if and only if α is a Poisson derivation on D and δ is a derivation on D such that

$$\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \delta(a)\alpha(b) - \alpha(a)\delta(b) \ \textit{ for all } a, b \in D. \tag{2}$$

The Poisson algebra $D[y]$ is denoted by $D[y; \alpha, \delta]$ and if δ is zero then it is denoted by $D[y; \alpha]$.

Proof.

$$(D, \{\text{-}, \text{-}\}) \underset{(\alpha \in \mathrm{PDer}(D), \delta \in \mathrm{Der}(D))\ (2)}{\overset{\alpha,\ \delta}{\rightleftarrows}} (D[y], (1)) \qquad \boxed{D[y; \alpha, \delta]}$$

Lemma 9 [Oh2, Lemma 1.3]

Let D be a Poisson algebra over K, $c \in K, u \in D$ and α, β are Poisson derivations such that

$$\alpha\beta = \beta\alpha \quad \textit{and} \quad \{d, u\} = (\alpha + \beta)(d)u \quad \textit{for all} \;\; d \in D \tag{3}$$

Then the polynomial ring $D[y, x]$ becomes a Poisson algebra with Poisson bracket

$$\{d, y\} = \alpha(d)y, \quad \{d, x\} = \beta(d)x \quad \textit{and} \quad \{y, x\} = cyx + u \tag{4}$$

for all $d \in D$. This Poisson algebra is denoted by $A = (D; \alpha, \beta, c, u)$ or $A = D[y; \alpha, 0][x; \beta, \delta' := u\frac{d}{dy}]$.

Proof.

By Theorem 8 By Theorem 8

$$(D, \{\text{-}, \text{-}\}) \xrightarrow{\alpha, \quad \delta = 0} (D[y], (1)) \xrightarrow[\delta' = u\frac{d}{dy}]{\beta, \quad \beta(y) = cy} (D[y][x], (4)) \qquad \boxed{(D; \alpha, \beta, c, u)}$$

$D[y; \alpha]$ $\quad$ $D[y; \alpha][x; \beta, \delta']$

The new Poisson algebra class $A = (K[t]; \alpha, \beta, c, u)$

We assume that

- K is an algebraically closed field with $\mathrm{char}(K) = 0$,
- D is a polynomial ring in one variable $K[t]$ with trivial Poisson bracket, i.e. $\{a, b\} = 0$ for all $a, b \in K[t]$, and
- $c \in K$.

If K-linear maps α and β are Poisson derivations on $K[t]$, i.e. $\alpha, \beta \in \mathrm{PDer}_K(K[t]) = \mathrm{Der}_K(K[t]) = K[t]\partial_t$ such that

$$\alpha = f\partial_t, \quad \beta = g\partial_t, \text{ where } \partial_t = \frac{d}{dt}, \text{ for some } f, g \in K[t].$$

Then by (3) in Lemma 9

$$0 = \{d, u\} = (\alpha + \beta)(d)u \text{ for all } d \in D$$

$$u \in \mathrm{PZ}(K[t]) = K[t].$$

$$(\alpha + \beta)u = 0.$$

Lemma 10

Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and α, β are in $\mathrm{Der}_K(K[t])$ *such that $\alpha = f\partial_t \neq 0$ and $\beta = g\partial_t \neq 0$ then*

$$\alpha\beta = \beta\alpha \ \text{ if and only if } \ g = \frac{1}{\lambda} f \ \text{ for some } \ \lambda \in K^{\times}.$$

This implies that precisely one of the three cases holds:

(Case I: $\alpha + \beta = 0$ and $u = 0$), (Case II: $\alpha + \beta = 0$ and $u \neq 0$) or (Case III: $\alpha + \beta \neq 0$ and $u = 0$) .

The classification of Poisson algebra class A, so far

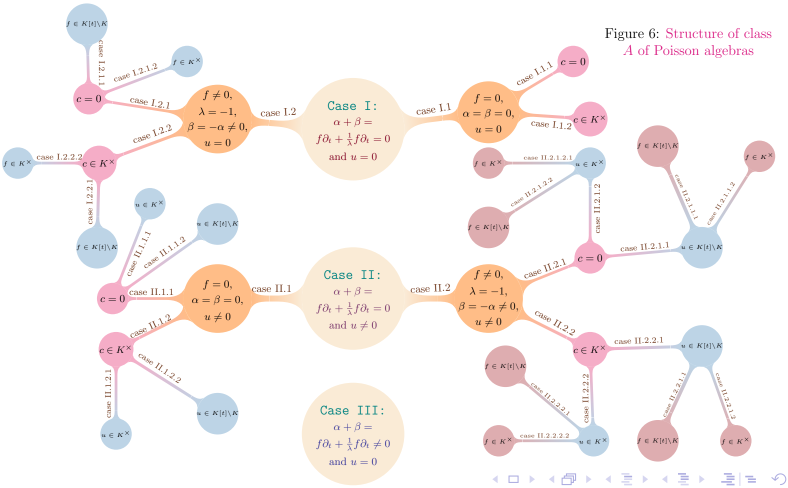

Figure 6: Structure of class A of Poisson algebras

The first case

Case I: $\alpha + \beta = f\partial_t + \frac{1}{\lambda} f\partial_t = (1 + \frac{1}{\lambda}) f\partial_t = 0$ and $u = 0$

Case I:

If $\alpha + \beta = f\partial_t + \frac{1}{\lambda} f\partial_t = 0$ for some $\lambda \in K^{\times}$, $f \in K[t]$, $u = 0$ and $c \in K$. Notice that, $f\partial_t + \frac{1}{\lambda} f\partial_t = 0$, implies that there are two subcases: $f = 0$ and $\lambda = -1$, see figure 7.

Case I.1:

If $f = 0$, i.e. $\alpha = \beta = 0$ and $u = 0$ then $A_1 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = 0, \ \ \{t, x\} = 0 \ \text{ and } \ \{y, x\} = cyx. \qquad (5)$$

There are two subcases: $c = 0$ and $c \in K^{\times}$.

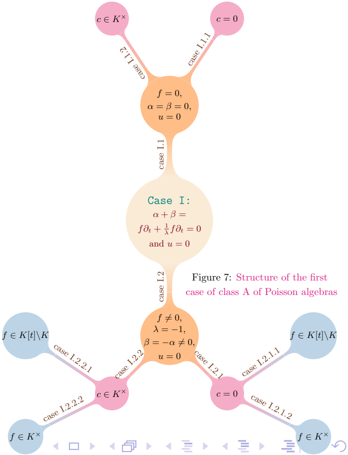

Figure 7: Structure of the first case of class A of Poisson algebras

Case I.1.1:

If in addition, $c = 0$ then the polynomial Poisson algebra $A_2 = (K[t]; 0, 0, 0, 0)$ has trivial Poisson structure and the Poisson spectrum of A_2 is the spectrum of a polynomial ring in three variables, i.e.

$$\mathrm{PSpec}(A_2) = \mathrm{Spec}(K[t, x, y]).$$

Case I.1.2:

If in addition, $c \in K^\times$ then $A_3 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket (5). In particular,

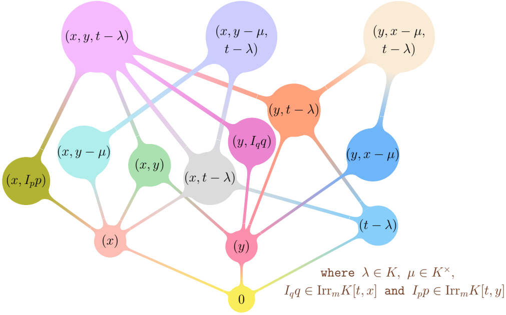

Figure 8: The containment information between Poisson prime ideals of A_3

$$A_3 = K[t] \otimes K[x, y]$$

is a tensor product of two Poisson algebras:

$$(K[t], \{\text{-}, \text{-}\} = 0) \quad \text{and} \quad (K[x, y], \{y, x\} = cyx).$$

Then we found $\mathrm{PSpec}(A_3)$, see figure 8.

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Thank you for listening

Further resources

Poisson algebra applications :

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- Poisson modules and Poisson enveloping algebras.

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Further resources

- Quantization and deformation algebras.

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- Semiclassical limits.

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