Introduction to Poisson geometry

Henrique Bursztyn, IMPA

Poisson school 2022, CRM

Basic intro with a view towards other talks this and next week....

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Plan of lectures:

- **Lectures 1 and 2**: Definition and examples of Poisson manifolds; basic properties.
- **Lecture 3**: Lie algebroids and symplectic groupoids.
- **Lecture 4**: Dirac structures (basics, applications...).

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◇ Weinstein, "Local structure of Poisson manifolds", JDG, 1983.

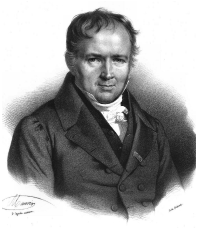
◇ Books by Cannas da Silva– Weinstein, Dufour–Zung, Pichereau–Laurent-Gengoux – Vanhaecke...

◇ New book: "Lectures on Poisson geometry", Crainic, Fernandes, Marcut.

Outline for lectures 1 and 2:

- "The" Poisson bracket.
- Definitions and (classes of) examples.
- Basic theory (local structure, symplectic foliation, some invariants).

Origins (19th century) and modern era (after 1970-80s)...



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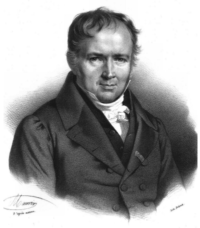

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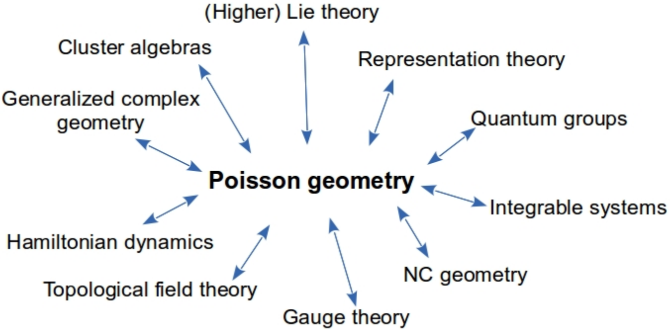

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Dynamical meaning of Poisson bracket: $\{H, f\} = \mathcal{L}_{X_H} f$

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S. Lie (1880): Lie algebras, Lie groups...

Modern era...

A. Lichnerowicz

A. Weinstein

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Why? Representation theory, Geometric mechanics (plasma physics), Deformation quantization...

Basic definitions

◇ A Poisson bracket on M is $\{\cdot,\cdot\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$:

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◇ Poisson map: $\phi : (M_1,\{\cdot,\cdot\}_1) \to (M_2,\{\cdot,\cdot\}_2)$,

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Poisson diffeos, also interesting weaker notions...

$\diamond$ Hamiltonian vector field of $f \in C^\infty(M)$: $X_f = \{f, \cdot\}$,

$$\mathcal{L}_{X_f} g = \{f, g\} = -\mathcal{L}_{X_g} f$$

If $X_f = 0$ we say that f is a *Casimir*.

Properties:

- $\mathcal{L}_{X_f} f = \{f, f\} = 0$,
- $\mathcal{L}_{X_H} f = 0$, $\mathcal{L}_{X_H} g = 0 \implies \mathcal{L}_{X_H}\{f, g\} = 0$,
- $[X_f, X_g] = X_{\{f,g\}}$.

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The Poisson bracket is example of a Poisson structure in $\mathbb{R}^{2n}$...

Tensorial viewpoint...

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◇ In local coordinates $(x_1, \ldots, x_n)$:

$$\{f, g\}(x) = \sum_{i,j} \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \qquad \pi_{ij} = \{x_i, x_j\},$$

$$\pi = \frac{1}{2} \sum_{i,j} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

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There exists $\Upsilon_\pi \in \mathfrak{X}^3(M)$ such that

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naturally described in terms of the *Schouten bracket* on $\mathfrak{X}^\bullet(M)$,

$$\Upsilon_\pi = \frac{1}{2}[\pi, \pi]$$

The Schouten bracket...

There is a unique $\mathbb{R}$-bilinear bracket

$$[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}$$

such that

- $[X, Y] = -(-1)^{(x-1)(y-1)}[Y, X]$,
- $[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(x-1)y} Y \wedge [X, Z]$
- For X vector field, $[X, \cdot] = \mathcal{L}_X$

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It is "the Poisson bracket" on $T^*[1]M$.

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- Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$, or
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- $\diamond$ Quotients by symmetries: M/G
- $\diamond$ Dual of Lie algebras $=$ Linear Poisson structures.
- $\diamond$ Direct products $M_1 \times M_2$,

$$\{f,g\}(x_1,x_2) = \{f_{x_2}, g_{x_2}\}_1(x_1) + \{f_{x_1}, g_{x_1}\}_2(x_2)$$

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$\diamond$ Other "symplectic Lie algebroids" (E-symplectic): b^k, c, elliptic, scattering....

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- Many examples in holomorphic/algebraic category

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Nondegenerate case (symplectic): Around any point, π is "the" Poisson bracket

$$\sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

This is *Darboux's theorem*.

Weinstein's splitting theorem (1983): *Around any* $z_0 \in M$*, there is isomorphism from* (M, π) *to product*

$$(S, \pi_S) \times (N, \pi_N), \quad z_0 = (x_0, y_0),$$

where π_S *is* symplectic *and* $\pi_N|_{y_0} = 0$.

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In coordinates centered at $z_0 = 0$:

$$\pi = \underbrace{\sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}}_{\pi_S} + \sum_{i<j} \underbrace{\varphi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}}_{\pi_N},$$

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Recent new approaches and generalizations (Frejlich-Marcut, Bursztyn-Lima-Meinrenken)

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Globally: "Leaves" on (M, π) are equivalence classes of

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Collection of symplectic leaves is *symplectic foliation*.

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- $\mathfrak{g}^*$, coadjoint orbits
  - $so(3)^*$: $z\partial_x\partial_y + x\partial_y\partial_z + y\partial_z\partial_x$
  - $sl(2)^*$: $-z\partial_x\partial_y + x\partial_y\partial_z + y\partial_z\partial_x$
  - $sb(2)^*$: $x\partial_x\partial_z + y\partial_y\partial_z$

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- Poisson Lie groups and homogeneous spaces, e,g, $SU(2) = S^3$ and $\mathbb{C}P^1 = S^2$....

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- $M = S/G$ for hamiltonian G-space S
- $\mathfrak{g}^*$, coadjoint orbits
  - $so(3)^*$: $\quad z\partial_x\partial_y + x\partial_y\partial_z + y\partial_z\partial_x$
  - $sl(2)^*$: $\quad -z\partial_x\partial_y + x\partial_y\partial_z + y\partial_z\partial_x$
  - $sb(2)^*$: $\quad x\partial_x\partial_z + y\partial_y\partial_z$
- Poisson Lie groups and homogeneous spaces, e,g, $SU(2) = S^3$ and $\mathbb{C}P^1 = S^2$....
- Log-symplectic structures...

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Hence Poisson geometry brings together:

Symplectic geometry $\leftrightsquigarrow$ Foliations $\leftrightsquigarrow$ Lie theory

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- Conn's theorem (isotropy Lie algebra semisimple, compact)
- Geometric proof by Crainic – Fernandes
- Recent paper Fernandes – Marcut

Some Poisson invariants...

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Poisson cohomology: $H^\bullet_\pi(M)$,

$$\ldots \longrightarrow \mathfrak{X}^k(M) \stackrel{d_\pi=[\pi,\cdot]}{\longrightarrow} \mathfrak{X}^{k+1}(M) \longrightarrow \ldots$$

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  - $H^0_\pi(M) = \text{Casimirs}(M)$,
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Nontrivial e.g. for log-symplectic manifolds...

Plan of lectures:

- **Lectures 1 and 2**:
  - Definitions (Poisson brackets, bivectors, hamiltonian vfs...)
  - Examples
  - Local theory (splitting, transverse, isotropy Lie algebra)
  - symplectic foliation
  - Invariants (Poisson cohomology, modular class)
- **Lecture 3**: Lie algebroids and symplectic groupoids.
- **Lecture 4**: Dirac structures (basics, applications...).

Plan for today:

- Symplectic realizations
- Lie groupoids / Lie algebroids
- Poisson structures and Lie algebroids
- Symplectic groupoids

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Property: $(\ker(d\mu))^{\omega}$ is involutive/integrable (Libermann's thm).

From realizations to dual pairs

A (full) dual pair is

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This leads to a weaker notion of equivalence (Morita equivalence, Picard groups).

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◇ realization versus "integration"?

General picture

Lie algebras $\rightleftharpoons$ Lie groups

Poisson manifolds $\rightleftharpoons$ symplectic groupoids

(M, π) $\qquad$ $(\mathcal{G}, \omega) \rightrightarrows M$

Lie algebroids $\rightleftharpoons$ Lie groupoids

Interlude: Lie groupoids and algebroids

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- composition law: $\mathsf{s}(gh) = \mathsf{s}(h)$, $\mathsf{t}(gh) = \mathsf{t}(g)$;
- associativity law: $(gh)k = g(hk)$
- law of units: $\mathsf{s}(1_x) = \mathsf{t}(1_x) = x$, and $g1_{\mathsf{s}(g)} = 1_{\mathsf{t}(g)}g = g$
- law of inverses: $\mathsf{s}(g^{-1}) = \mathsf{t}(g)$, $\mathsf{t}(g^{-1}) = \mathsf{s}(g)$ and $g^{-1}g = 1_{\mathsf{s}(g)}, \quad gg^{-1} = 1_{\mathsf{t}(g)}$.

Some examples

◇ Lie groups

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- ◇ *b*- (a.k.a *log*-) tangent bundles ${}^b TM$ (a.k.a. $T_Z M$).

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E.g. A-Poisson structures...

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Lie theorems and the integration problem: not every Lie algebroid is integrable! (obstructions due to Crainic-Fernandes, 2003).

Poisson structures vs Lie algebroids

Poisson manifolds are Lie algebroids

The *cotangent Lie algebroid* of a Poisson manifold (M, π):

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Global counterparts?

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◇ Generically nondegenerate Poisson structures as symplectic Lie algebroids...

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Others: b^k, $c-$, elliptic, scattering...

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- ◇ Symplectic groupoid integrates $(T^*M)_\pi$, and ssc integration is symplectic.
- ◇ Complete realizations $\iff$ integrability
- ◇ Can we always "integrate" a Poisson manifold? No...

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Other aspects, applications...

◇ Poisson homotopy groupoid and Poisson sigma model.

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- $\diamond$ Poisson homotopy groupoid and Poisson sigma model.
- $\diamond$ Symplectic groupoids and quantization
- $\diamond$ Symplectic groupoids and moment maps
- $\diamond$ Symplectic groupoid as tool in Poisson geometry:
  - calculation of invariants, vanishing of cohomologies... e.g. geometric proof of Conn's linearization.
  - Poisson manifolds of compact types

Other aspects, applications...

- ◇ Poisson homotopy groupoid and Poisson sigma model.
- ◇ Symplectic groupoids and quantization
- ◇ Symplectic groupoids and moment maps
- ◇ Symplectic groupoid as tool in Poisson geometry:
  - calculation of invariants, vanishing of cohomologies... e.g. geometric proof of Conn's linearization.
  - Poisson manifolds of compact types
- ◇ Poisson groupoids; Lie bialgebroids and Courant algebroids.

Plan of lectures:

- **Lectures 1 and 2**:
  - Definitions (Poisson brackets, bivectors, hamiltonian vfs...)
  - Examples
  - Local theory (splitting, transverse, isotropy Lie algebra)
  - symplectic foliation
  - Invariants (Poisson cohomology, modular class)
- **Lecture 3**: Lie algebroids and symplectic groupoids.
  - Symplectic realizations
  - Lie groupoids, Lie algebroids
  - Lie algebroids in Poisson geometry
  - Symplectic groupoids
- **Lecture 4**: Dirac structures (basics, applications...).

Plan for today:

- Dirac structures
  - Motivation
  - Definition, examples, properties...
  - Applications

From symplectic geometry to Dirac structures

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Dirac's brackets for "second class" $C = \{x,\, \varphi^i(x) = 0\}$:

$$\{f, g\}_{Dirac} := (\{F, G\} - \{F, \varphi^i\} c_{ij} \{\varphi^j, G\})|_C,$$

where $c^{ij} = \{\varphi^i, \varphi^j\}$.

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what do they have in common?

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Prototypical example of a *Courant algebroid* $(E, \rho, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$

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For $z \neq 0$, this is graph of $\pi = \frac{1}{z}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$:

$$\{x, y\} = \frac{1}{z}, \ \ \{x, z\} = 0, \ \ \{y, z\} = 0.$$

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Caveat: Transversality/cleaness issues...

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Role of Dirac maps in reduction.

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Then

- Dirac structures: modified integrability conditions, but similar properties...
- Twisted Poisson structure: $\frac{1}{2}[\pi,\pi] = \pi^\sharp(\phi)$

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Singular foliation: Conjugacy classes

Leafwise 2-form (G.H.J.W. '97):

$$\omega(u_G, v_G)|_g := \left\langle \frac{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}{2} u, v \right\rangle_{\mathfrak{g}}$$

Applications....

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Other aspects: volume forms, pure spinors (Alekseev, B.-, Meinrenken), q-Poisson...

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$\diamond$ More general complex Dirac structures? (D. Aguero, R. Rubio)

Other applications:

- Normal forms around transversals (for Poisson, GC, etc.).
- Poisson homogeneous spaces (classification and integration).
- Super/graded geometric approach.
- Shifted symplectic structures

Thanks for your attention!