Introduction in Generalized Weyl Poisson algebras

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A nonempty set S' with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set S with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. S is an abelian if $g + h = h + g$.

A nonempty set V with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field K if for all $\lambda_1, \lambda_2 \in K$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
6. $1 \times v = v$.

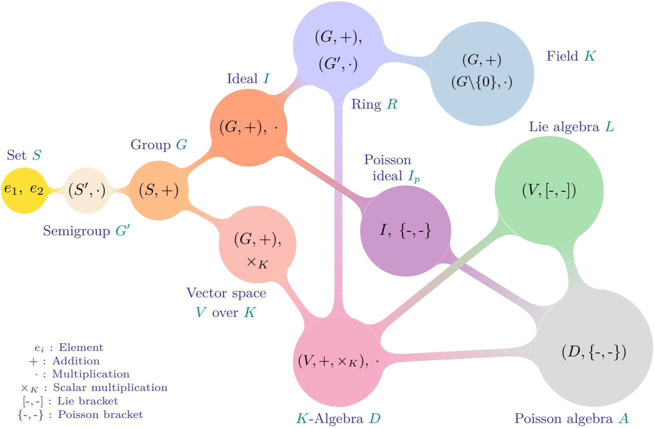

Figure 1: Algebraic structure

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Definition 1

A vector space L over a field K is called a *Lie algebra* if there exists a bilinear product $[\text{-},\text{-}]$ on L, called a *Lie bracket*, which is *anti-commutative* and satisfies the *Jacobi identity*: $[a,b] = -[b,a]$ and

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0 \text{ for all } a,b,c \in L.$$

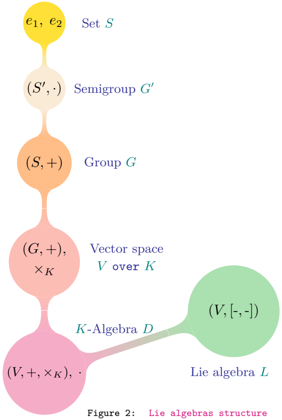

Figure 2: Lie algebras structure

The n'th Weyl algebra

Definition 2

Let K be a field. An associative K-algebra A_n that is generated by $2n$ elements $X_1, \dots, X_n, Y_1, \dots, Y_n$, subject to the relations:

$$[Y_i, X_j] = \delta_{ij} \text{ and } [X_i, X_j] = [Y_i, Y_j] = 0 \text{ for all } i, j,$$

is called the *n'th Weyl algebra* $A_n(K)$, where δ_{ij} is the Kronecker delta function, i.e.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Definition 3 [Bav1]

Let R be a ring, $\sigma = (\sigma_1, \dots, \sigma_n)$ be an n-tuple of commuting automorphisms of R, $a = (a_1, \dots, a_n)$ be an n-tuple of elements in the centre $Z(R)$, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$. The *generalized Weyl algebra* (GWA) $\mathcal{A} = R[X, Y; \sigma, a]$ of rank n is a ring generated by R and $2n$ elements $X_1, \dots, X_n, Y_1, \dots, Y_n$ subject to the defining relations:

$$Y_i X_i = a_i, \quad X_i Y_i = \sigma_i(a_i),$$

$$X_i d = \sigma_i(d) X_i, \; Y_i d = \sigma_i^{-1}(d) Y_i \; \text{ for all } d \in R,$$

$$[X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0 \; \text{ for all } i \neq j,$$

where $[x, y] = xy - yx$.

Example 4 [Bav1]

The Weyl algebra A_n is a GWA $\mathcal{A} = R[X, Y; \sigma, a]$ of rank n, where $R = K[H_1, \dots, H_n]$ is a polynomial ring in n variables, $\sigma = (\sigma_1, \dots, \sigma_n)$ such that

$$\sigma_i(H_j) = H_j - \delta_{ij} \text{ and } \quad a = (H_1, \dots, H_n),$$

where δ_{ij} is the Kronecker delta function. The map

$$A_n \to \mathcal{A}, \;\; X_i \mapsto X_i, \;\; Y_i \mapsto Y_i, \;\; \text{and} \;\; Y_i X_i \mapsto H_i \quad \text{for all} \;\; i = 1, \dots, n$$

is an algebra isomorphism.

Definition 5

A (commutative) K-algebra $(D,\ +,\ \cdot)$ is called a *Poisson algebra* if there exists bilinear product $\{\text{-},\text{-}\}$ on D, called a Poisson bracket, such that $(D, \{\text{-},\text{-}\})$ is a Lie algebra and Leibniz rule holds, i.e.

$$\{a \cdot b, c\} = \{a, c\} \cdot b + a \cdot \{b, c\} \text{ for all } a, b, c \in D.$$

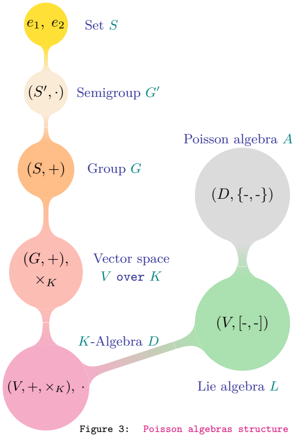

Figure 3: Poisson algebras structure

Definition 6 [Bav2]

Let D be a Poisson algebra, $\partial = (\partial_1, \ldots, \partial_n) \in \mathrm{PDer}_K(D)^n$ be an n-tuple of commuting Poisson derivations of D, $a = (a_1, \ldots, a_n) \in \mathrm{PZ}(D)^n$ be an n-tuple of Poisson central elements of D such that $\partial_i(a_j) = 0$ for all $i \neq j$. The commutative GWA

$$A = D[X, Y; a] = D[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1 - a_1, \ldots, X_nY_n - a_n)$$

is a Poisson algebra with Poisson bracket defined by the rule: For all $i, j = 1, \ldots, n$ and $d \in D$,

$$\{Y_i, d\} = \partial_i(d)Y_i, \ \ \{X_i, d\} = -\partial_i(d)X_i, \ \ \{Y_i, X_i\} = \partial_i(a_i), \ \text{ and}$$
$$\{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0 \ \text{ for all } \ i \neq j.$$

This Poisson algebra is denoted by $A = D[X, Y; a, \partial\}$ and is called the *generalized Weyl Poisson algebra* of rank n (GWPA) where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$.

Lemma 7 [Bav2]

Let $A = D[X, Y; a, \partial\}$ be a GWPA of rank n. Let $\mathcal{A} = D[X, Y; \partial, \partial(a)]$ be a Poisson algebra, $\partial(a) = (\partial_1(a_1), \dots, \partial_n(a_n))$ with Poisson bracket,

$$\{Y_i, d\} = \partial_i(d)Y_i, \;\; \{X_i, d\} = -\partial_i(d)X_i, \;\; \{Y_i, X_i\} = \partial_i(a_i), \;\; \text{and}$$
$$\{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0 \;\; \text{for all} \;\; i \neq j.$$

Then the elements $X_1Y_1 - a_1, \dots, X_nY_n - a_n \in \mathrm{PZ}(\mathcal{A})$ and the GWPA $A = D[X, Y; a, \partial\}$ is a factor algebra of the Poisson algebra $\mathcal{A}$,

$$A \cong \mathcal{A}/(X_1Y_1 - a_1, \dots, X_nY_n - a_n).$$

Example 8 [Bav2]

The classical Poisson polynomial algebra $P_{2n} = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ with Poisson bracket $\{Y_i, X_j\} = \delta_{ij}$ and $\{X_i, X_j\} = \{Y_i, Y_j\} = 0$ for all i, j, where δ_{ij} is the Kronecker delta function, is a GWPA

$$P_{2n} \cong K[H_1, \ldots, H_n][X, Y; a, \partial\},$$

where $K[H_1, \ldots, H_n]$ is a Poisson polynomial algebra with trivial Poisson bracket, $a = (H_1, \ldots, H_n)$, $\partial = (\partial_1, \ldots, \partial_n)$ and $\partial_i = \frac{\partial}{\partial H_i}$ (via the isomorphism of Poisson algebras

$$P_{2n} \to K[H_1, \ldots, H_n][X, Y; a, \partial\}, \;\; X_i \mapsto X_i, \;\; Y_i \mapsto Y_i, \; X_iY_i \mapsto H_i)$$

References

[Bav1] V. V. Bavula, Generalized Weyl algebras and their representations. *St. Petersburg Math. J*, **4** (1993), no. 1, 71–92.

[Bav2] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

[GoWa] K. R. Goodearl and R. B. Warfield. *An introduction to noncommutative noetherian rings*. 2nd ed. New York: Cambridge University Press. (2004), pages 1–85, 105–122 and 166–186.

[Oh2] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265–1277.

Thank you for listening

Further Reading

Poisson algebra applications:

- Skew polynomial rings

[MyOh] N.-H. Myung and S.-Q. Oh, A construction of an Iterated Ore extension. arXiv:1707.05160v2 (2018).

- Poisson polynomial ring

[Oh2] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265–1277.

- Poisson modules and Poisson enveloping algebras

[Oh1] S.-Q. Oh, Poisson enveloping algebras. *Communications in Algebra*, **27** (1999), 2181–2186.

- Generalised Weyl algebras and generalised Weyl Poisson algebras

[Bav1] V. V. Bavula, Generalized Weyl algebras and their representations. *St. Petersburg Math. J*, **4** (1993), no. 1, 71–92.

[Bav2] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

Further Reading

- Quantization and deformation algebras

[BPR] C. Beem, W. Peelaers and L. Rastelli, Deformation quantization and superconformal symmetry in three dimensions. *Comm. Math. Phys.* **354** (2017), 345–392.

- Semiclassical limits

[ChOh] E.-H. Cho and S.-Q. Oh, Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra. *Lett. Math. Phys.* **106** (2016), 997–1009.

- Star product and Poisson algebras

[EtSt] P. Etingof and D. Stryker, Short star-products for filtered quantizations, I. arXiv:1909.13588v5 (2020).

- Poisson manifolds.

[Kon] M. Kontsevich, Deformation quantization of Poisson manifolds. arXiv:q-alg/9709040v1 (1997).

- Hamiltonian systems.