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POISSON POLYNOMIAL RINGS

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Let A be a Poisson algebra with Poisson bracket $\{\cdot,\cdot\}_A$ and let α, δ be linear maps from A into itself. Here we find a necessary and sufficient condition for the pair (α, δ) such that the polynomial ring $A[x]$ has the Poisson bracket

$$\{a, b\} = \{a, b\}_A, \qquad \{a, x\} = \alpha(a)x + \delta(a)$$

for all $a, b \in A$ and construct a class of Poisson algebras including the coordinate rings of Poisson 2×2-matrices and Poisson symplectic 4-space.

Key Words: Derivation; Poisson algebra.

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Many quantum groups have been constructed from Poisson algebras which are polynomial rings with certain Poisson brackets. In this article, we find that many Poisson brackets on polynomial rings are given by derivations with certain conditions, which may be considered as a Poisson version of a skew polynomial ring constructed by an endomorphism α and an α-derivation. Moreover a Poisson structure of a class of Poisson algebras including the coordinate rings of Poisson 2×2-matrices and Poisson symplectic 4-space is investigated. Assume throughout the article that $\mathbf{k}$ is a field and that all Poisson algebras are commutative and finitely generated as a $\mathbf{k}$-algebra.

1. POISSON POLYNOMIAL RINGS

1.1. Let A be a Poisson algebra over $\mathbf{k}$. A $\mathbf{k}$-linear map $\alpha : A \longrightarrow A$ is said to be a derivation (respectively Poisson derivation) on A if it satisfies (i) (respectively (i) and (ii)) of the following conditions (for all $a, b \in A$):

(i) $\alpha(ab) = \alpha(a)b + a\alpha(b)$;
(ii) $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$.

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Theorem. *Let α, δ be* $\mathbf{k}$*-linear maps on a Poisson algebra A with Poisson bracket $\{\cdot,\cdot\}_A$. Then the polynomial ring $A[x]$ becomes a Poisson algebra with Poisson bracket*

$$\{a, b\} = \{a, b\}_A, \qquad \{a, x\} = \alpha(a)x + \delta(a) \tag{1.1}$$

for all $a, b \in A$ if and only if α is a Poisson derivation and δ is a derivation such that

$$\delta(\{a, b\}_A) - \{\delta(a), b\}_A - \{a, \delta(b)\}_A = \delta(a)\alpha(b) - \alpha(a)\delta(b) \tag{1.2}$$

for all $a, b \in A$. In this case, we denote the Poisson algebra $A[x]$ by $A[x; \alpha, \delta]_p$ and if $\delta = 0$, then we simply write $A[x; \alpha]_p$ for $A[x; \alpha, 0]_p$.

Proof. If $A[x]$ is a Poisson algebra with the Poisson bracket (1.1) then we have that

$$\begin{aligned} \{ab, x\} &= \alpha(ab)x + \delta(ab) \\ a\{b, x\} + \{a, x\}b &= (a\alpha(b) + \alpha(a)b)x + (a\delta(b) + \delta(a)b) \end{aligned}$$

for all $a, b \in A$, and thus both α and δ are derivations on A. Moreover, since the Poisson bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity, we have that

$$\begin{aligned} 0 &= \{\{a, b\}, x\} + \{\{b, x\}, a\} + \{\{x, a\}, b\} \\ &= (\alpha(\{a, b\}_A) - \{\alpha(a), b\}_A - \{a, \alpha(b)\}_A)x \\ &\quad + \delta(\{a, b\}_A) - \{\delta(a), b\}_A - \{a, \delta(b)\}_A - \delta(a)\alpha(b) + \alpha(a)\delta(b) \end{aligned}$$

for all $a, b \in A$. Hence α is a Poisson derivation and δ is a derivation such that the pair (α, δ) satisfies (1.2).

Conversely, we suppose that α is a Poisson derivation and δ is a derivation such that (α, δ) satisfies (1.2). Define a $\mathbf{k}$-bilinear map $\{\cdot,\cdot\} : A[x] \times A[x] \longrightarrow A[x]$ by

$$\{ax^i, bx^j\} = (\{a, b\}_A + jb\alpha(a) - ia\alpha(b))x^{i+j} + (jb\delta(a) - ia\delta(b))x^{i+j-1} \tag{1.3}$$

for all monomials ax^i and bx^j in $A[x]$. Note that the case for $i = 0$, $j = 1$, and $b = 1$ in (1.3) is (1.1). Then, by (1.3), we have that $\{f, g\} = -\{g, f\}$ for all $f, g \in A[x]$ and that, for a fixed element $f \in A[x]$, $\mathbf{k}$-linear maps

$$\begin{aligned} \{f, \cdot\} &: A[x] \longrightarrow A[x], \qquad g \mapsto \{f, g\} \\ \{\cdot, f\} &: A[x] \longrightarrow A[x], \qquad g \mapsto \{g, f\} \end{aligned}$$

are derivations on $A[x]$ since α and δ are derivations. It remains to check the Jacobi identity: for $ax^i, bx^j, cx^k \in A[x]$,

$$\left\{\left\{ax^i, bx^j\right\}, cx^k\right\} + \left\{\left\{bx^j, cx^k\right\}, ax^i\right\} + \left\{\left\{cx^k, ax^i\right\}, bx^j\right\} = 0. \tag{1.4}$$

We proceed by the induction on i, j, k. The case $i = j = k = 0$ in (1.4) is trivial. The case $i = 1, j = k = 0, a = 1$ is shown immediately by (1.2). Suppose that the case for $j = k = 0$ in (1.4) holds and we check the case for $i + 1$, $j = k = 0$:

$$\begin{aligned}&\{\{ax^{i+1}, b\}, c\} + \{\{b, c\}, ax^{i+1}\} + \{\{c, ax^{i+1}\}, b\}\\&= (\{\{ax^{i}, b\}, c\} + \{\{b, c\}, ax^{i}\} + \{\{c, ax^{i}\}, b\})x\\&\quad + ax^{i}(\{\{x, b\}, c\} + \{\{b, c\}, x\} + \{\{c, x\}, b\})\\&= 0,\end{aligned}$$

by the Leibniz rule and the induction hypothesis. Hence (1.4) for the case $j = k = 0$ holds. Now check (1.4) for the case $k = 0$ using induction on j and the general case using induction on k. The proof is complete. □

1.2 Lemma. *Let α, δ be derivations on a Poisson algebra A generated by a set X as an algebra.*

(i) *If $\alpha(a) = \delta(a)$ for all $a \in X$, then $\alpha = \delta$.*
(ii) *If $\alpha\delta(a) = \delta\alpha(a)$ for all $a \in X$, then $\alpha\delta = \delta\alpha$.*
(iii) *If α satisfies $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in X$, then α is a Poisson derivation.*
(iv) *If α and δ satisfy* (1.2) *for all elements in X, then α and δ satisfy* (1.2) *for all elements in A.*

Proof. It is proven by a straightforward computation combined with induction. □

1.3 Lemma. *Let $c \in \mathbf{k}$, $u \in A$ and let α, β be Poisson derivations on A such that*

$$\alpha\beta = \beta\alpha, \qquad \{a, u\} = (\alpha + \beta)(a)u \tag{1.5}$$

for all $a \in A$. Then the polynomial ring $A[y, x]$ becomes a Poisson algebra with Poisson bracket

$$\{a, y\} = \alpha(a)y, \qquad \{a, x\} = \beta(a)x, \qquad \{y, x\} = cyx + u \tag{1.6}$$

for all $a \in A$.

The Poisson algebra $A[y, x]$ with Poisson bracket (1.6) *is denoted by $(A; \alpha, \beta, c, u)$.*

Proof. By 1.1, there exists a Poisson algebra $A[y; \alpha]_p$ with Poisson bracket $\{a, y\} = \alpha(a)y$ for all $a \in A$. The derivation β is extended to a derivation, still denoted by β, to $A[y; \alpha]_p$ by setting $\beta(y) = cy$ and $\delta = u\frac{d}{dy}$ is a derivation on $A[y; \alpha]_p$ satisfying $\delta(y) = u$ and $\delta(a) = 0$ for all $a \in A$. Let us prove that, for all $f, g \in A[y; \alpha]_p$,

$$\begin{aligned}&\beta(\{f, g\}) = \{\beta(f), g\} + \{f, \beta(g)\}\\&\delta(\{f, g\}) = \{\delta(f), g\} + \{f, \delta(g)\} + \delta(f)\beta(g) - \beta(f)\delta(g).\end{aligned} \tag{1.7}$$

If $f, g \in A$ then (1.7) holds trivially. Hence it is enough to prove (1.7) for the case $f = a \in A$ and $g = y$ by 1.2. Now

$$\begin{aligned}\beta(\{a, y\}) &= \beta(\alpha(a)y) = \alpha(a)\beta(y) + \beta(\alpha(a))y \\ &= c\alpha(a)y + \alpha(\beta(a))y = \{\beta(a), y\} + \{a, \beta(y)\} \\ \delta(\{a, y\}) &= \delta(\alpha(a)y) = \alpha(a)u = \{a, u\} - \beta(a)u \\ &= \{\delta(a), y\} + \{a, \delta(y)\} + \delta(a)\beta(y) - \beta(a)\delta(y),\end{aligned}$$

as claimed. Therefore β is a Poisson derivation on $A[y; \alpha]_p$ such that the pair (β, δ) satisfies (1.2). It follows that, by 1.1, there exists the Poisson algebra $A[y, x] = A[y; \alpha]_p[x; \beta, \delta]_p$ with Poisson bracket (1.6). □

2. EXAMPLES

2.1 Poisson *k*[*y*, *x*]. Let $\mathbf{k}[y]$ be a polynomial ring. Note that $\mathbf{k}[y]$ is a Poisson algebra with trivial Poisson bracket. For any $f, g \in \mathbf{k}[y]$, set

$$\alpha = f\frac{d}{dy}, \qquad \delta = g\frac{d}{dy}.$$

Then α is a Poisson derivation, δ is a derivation and (α, δ) satisfies (1.2) clearly. Hence, by 1.1, $\mathbf{k}[y, x] = \mathbf{k}[y][x; \alpha, \delta]_p$ has the Poisson bracket

$$\{y, x\} = fx + g.$$

2.2 Poisson *n*-Space. Let $\Lambda = (\lambda_{ij})$ be a skew-symmetric $n \times n$-matrix. Note that $\mathbf{k}[x_1]$ is a Poisson algebra with trivial Poisson bracket. For each $i = 2, \ldots, n$, denote the derivation $\lambda_{1i}x_1\frac{\partial}{\partial x_1} + \cdots + \lambda_{i-1,i}x_{i-1}\frac{\partial}{\partial x_{i-1}}$ on $\mathbf{k}[x_1, \ldots, x_{i-1}]$ by α_i. Since α_2 is a Poisson derivation on $\mathbf{k}[x_1]$, there exists a Poisson algebra $\mathbf{k}[x_1][x_2; \alpha_2]$. Suppose that there exists a Poisson algebra $B = \mathbf{k}[x_1][x_2; \alpha_2]_p \cdots [x_{n-1}; \alpha_{n-1}]_p$. It is checked using 1.2 that α_n is a Poisson derivation on B, and hence there exists a Poisson algebra $B[x_n; \alpha_n]_p = \mathbf{k}[x_1, \ldots, x_n]$ by 1.1. Therefore, by induction on n, the coordinate ring $\mathcal{O}(\mathbf{k}^n) = \mathbf{k}[x_1, \ldots, x_n]$ of n-space $\mathbf{k}^n$ is a Poisson algebra with Poisson bracket

$$\{x_i, x_j\} = \lambda_{ij}x_ix_j$$

for all i, j. (See Oh, 1999, §2.)

2.3 Poisson 2*n*-Space. The polynomial ring $A = \mathbf{k}[y_1, x_1, \ldots, y_n, x_n]$ has the Poisson bracket defined by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial y_i}\frac{\partial f}{\partial x_i}$$

for all $f, g \in A$, which is given in Chari and Pressley (1994, p. 18). Setting A_i to be the Poisson subalgebra of A generated by $y_1, x_1, \ldots, y_i, x_i$, A_i can be presented by

$$A_i = A_{i-1}[y_i; 0]_p[x_i; 0, \delta_i]_p = (A_{i-1}; 0, 0, 0, 1),$$

where δ_i is defined by

$$\delta_i(y_j) = 0, \qquad \delta_i(x_j) = 0, \qquad \delta_i(y_i) = 1$$

for all $j = 1, \ldots, i-1$.

2.4 Poisson $M_2(\mathbf{k})$. For the Poisson algebra $\mathbf{k}[b, c]$ with trivial Poisson bracket, that is, $\{b, c\} = 0$, the derivation $\alpha = -2b\frac{\partial}{\partial b} - 2c\frac{\partial}{\partial c}$ on $\mathbf{k}[b, c]$ is clearly a Poisson derivation. By 1.3, there exists a Poisson algebra

$$(\mathbf{k}[b, c]; \alpha, -\alpha, 0, 4bc)$$

which is the Poisson algebra $\mathcal{O}(M_2(\mathbf{k})) = \mathbf{k}[b, c][a, d]$ with Poisson bracket

$$\{b, c\} = 0, \qquad \{b, a\} = -2ba, \qquad \{c, a\} = -2ca,$$
$$\{b, d\} = 2bd, \qquad \{c, d\} = 2cd, \qquad \{a, d\} = 4bc,$$

given in Oh (1999, 2.9), Korogodski and Soibelman (1998, Example 3.2.9), and Vancliff (1999, 3.13).

2.5 Generalization of 2.3 and 2.4. Let α be a Poisson derivation on a Poisson algebra A, and let u be a central Poisson element of A, that is, $\{a, u\} = 0$ for all $a \in A$. Then we have a Poisson algebra $(A; \alpha, -\alpha, c, u)$ for any $c \in \mathbf{k}$ since $\alpha, \beta = -\alpha$ and u satisfy (1.5). Observe that $(A; \alpha, -\alpha, c, u)$ is a generalized form of 2.3 and 2.4.

2.6 Poisson $M_n(\mathbf{k})$. It is well known that the coordinate ring $\mathcal{O}(M_n(\mathbf{k})) = \mathbf{k}[x_{ij} \mid i, j = 1, \ldots, n]$ of $n \times n$ matrices $M_n(\mathbf{k})$ is a Poisson algebra with Poisson bracket

$$\{x_{ij}, x_{rs}\} = \begin{cases} 2x_{ij}x_{rs} & i = r, \quad j < s \\ 2x_{ij}x_{rs} & i < r, \quad j = s \\ 0 & i < r, \quad j > s \\ 4x_{is}x_{rj} & i < r, \quad j < s. \end{cases}$$

That is, each subalgebra of $\mathcal{O}(M_n(\mathbf{k}))$ generated by four generators

$$\begin{matrix} x_{ij} & x_{is} \\ x_{rj} & x_{rs} \end{matrix}$$

is equal to the Poisson algebra $\mathcal{O}(M_2(\mathbf{k}))$. Observe that $\mathcal{O}(M_n(\mathbf{k}))$ can be presented by an iterated Poisson polynomial ring

$$\mathbf{k}[x_{11}][x_{12}; \alpha_{12}, \delta_{12}]_p[x_{21}; \alpha_{21}, \delta_{21}]_p[x_{13}; \alpha_{13}, \delta_{13}]_p[x_{22}; \alpha_{22}, \delta_{22}]_p \cdots [x_{nn}; \alpha_{nn}, \delta_{nn}]_p,$$

where α_{rs}, δ_{rs} are defined by

$$\alpha_{rs}(x_{ij}) = \begin{cases} 2x_{ij} & i = r, \quad j < s \\ 2x_{ij} & i < r, \quad j = s \\ 0 & i < r, \quad j > s \\ 0 & i < r, \quad j < s \end{cases}$$

$$\delta_{rs}(x_{ij}) = \begin{cases} 0 & i = r, \quad j < s \\ 0 & i < r, \quad j = s \\ 0 & i < r, \quad j > s \\ 4x_{is}x_{rj} & i < r, \quad j < s. \end{cases}$$

2.7 Poisson Symplectic and Euclidean 4-Spaces. Let

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} \\ -\gamma_{21} & 0 \end{pmatrix}, \qquad P = (p_1, p_2), \qquad Q = (q_1, q_2),$$

where $\gamma_{12}, p_1, p_2, q_1, q_2 \in \mathbf{k}$ such that $p_1 \neq q_1, p_2 \neq q_2$. Since $\tau = -q_1 y_1 \frac{\partial}{\partial y_1}$ is a Poisson derivation on $\mathbf{k}[y_1]$, there exists a Poisson algebra $A = \mathbf{k}[y_1][x_1; \tau]_p$ by 1.1. Define derivations α, β on A by

$$\alpha = \gamma_{12} y_1 \frac{\partial}{\partial y_1} + (p_2 - \gamma_{12}) x_1 \frac{\partial}{\partial x_1}$$
$$\beta = -(q_1 + \gamma_{12}) y_1 \frac{\partial}{\partial y_1} + (q_1 - p_2 + \gamma_{12}) x_1 \frac{\partial}{\partial x_1}.$$

It is easy to check using 1.2 that α and β are Poisson derivations on A such that $\alpha\beta = \beta\alpha$. Setting $u = -(q_1 - p_1) y_1 x_1 \in A$, the triple (α, β, u) satisfies (1.5), and thus there exists a Poisson algebra $A_{2,\Gamma}^{P,Q} = (A; \alpha, \beta, -q_2, u)$ by 1.3. That is, $A_{2,\Gamma}^{P,Q} = \mathbf{k}[y_1, x_1, y_2, x_2]$ has the following Poisson bracket:

$$\begin{aligned}
\{y_1, y_2\} &= \gamma_{12} y_1 y_2 \\
\{x_1, y_2\} &= (p_2 - \gamma_{12}) x_1 y_2 \\
\{y_1, x_2\} &= -(q_1 + \gamma_{12}) y_1 x_2 \\
\{x_1, x_2\} &= (q_1 - p_2 + \gamma_{12}) x_1 x_2 \\
\{x_1, y_1\} &= q_1 y_1 x_1 \\
\{x_2, y_2\} &= q_2 y_2 x_2 + (q_1 - p_1) y_1 x_1.
\end{aligned}$$

The Poisson algebra $A_{2,\Gamma}^{P,Q}$ is said to be the coordinate ring of Poisson symplectic (respectively Euclidean) 4-space if

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P = (0, 0), \qquad Q = (-2, -2)$$

$$\left(\text{respectively } \Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P = (-2, -2), \quad Q = (0, 0)\right).$$

Observe that the coordinate ring of Poisson Euclidean 4-space is Poisson isomorphic to $\mathscr{O}(M_2(\mathbf{k}))$ given in 2.4.

3. PRIME POISSON IDEALS OF $A[x; \alpha]_p$

Here we develop a technique to find prime Poisson ideals of Poisson polynomial rings and classify the prime Poisson ideals of $A_{2,\Gamma}^{P,Q}$ given in 2.7.

3.1 Definition.

(i) An ideal I of a Poisson algebra A is said to be a Poisson ideal of A if $\{I, A\} \subseteq I$. A Poisson ideal P is said to be a prime Poisson ideal if it is a prime ideal.
(ii) A Poisson derivation α on a Poisson algebra A is said to be inner if there exists an invertible element $a \in A$ such that $\alpha(b) = a^{-1}\{b, a\}$ for all $b \in A$.
(iii) Let A be a Poisson algebra and let Λ be a set of linear maps from A into itself. An ideal (respectively Poisson ideal) I of A is said to be a Λ-ideal (respectively Λ-Poisson ideal) or Λ-stable if $\lambda(I) \subseteq I$ for all $\lambda \in \Lambda$.
(iv) Let α be a Poisson derivation on a Poisson algebra. A Poisson algebra is said to be Poisson simple (respectively α-Poisson simple) if it has no nontrivial Poisson ideal (respectively α-Poisson ideal).
(v) An element a of a Poisson algebra A is said to be Poisson normal if $\{a, A\} \subseteq aA$.
(vi) Let α be a Poisson derivation of a Poisson algebra A. An α-ideal P of A is said to be an α-prime Poisson ideal if it is a prime Poisson ideal.

3.2 Lemma. (i) *Let S be a multiplicative subset of a Poisson algebra A. Then any derivation on A is uniquely extended to $S^{-1}A$ and*

$$(S^{-1}A)[x; \alpha', \delta']_p \cong S^{-1}(A[x; \alpha, \delta]_p),$$

where α', δ' are the extensions of α, δ on $S^{-1}A$, respectively.

(ii) *Let J be a Poisson ideal of $B = A[x; \alpha, \delta]_p$ and, for each non-negative integer n, let J_n be the set of leading coefficients of polynomials in J with degree n, together with zero. Then each J_n is a Poisson ideal of A with $J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$ and $\alpha(J_n) \subseteq J_{n+1}$. In particular, the set of leading coefficients of polynomials in J, together with zero, is an α-Poisson ideal of A.*

(iii) *Let I be an (α, δ)-Poisson ideal of A. Then $IA[x; \alpha, \delta]_p$ is a Poisson ideal of $A[x; \alpha, \delta]_p$ and*

$$A[x; \alpha, \delta]_p / IA[x; \alpha, \delta]_p \cong (A/I)[x; \bar{\alpha}, \bar{\delta}]_p,$$

where $\bar{\alpha}$ and $\bar{\delta}$ are the maps induced by α and δ, respectively.

(iv) *A Poisson algebra $A[x; \alpha]_p[y; \beta]_p$ such that $\beta(A) \subseteq A$ and $\beta(x) = bx$ for some element $b \in A$ is equal to $A[y; \beta']_p[x; \alpha']_p$, where $\beta' = \beta|_A$ and $\alpha' : A[y; \beta']_p \longrightarrow A[y; \beta']_p$ is defined by*

$$\alpha'(a) = \alpha(a), \qquad \alpha'(y) = -by$$

for all $a \in A$.

Proof. (i) McConnell and Robson (1987, 15.1.23).

(ii) Clearly, J_n is a Poisson ideal and $\alpha(J_n) \subseteq J_{n+1}$. Hence the set J' of leading coefficients of polynomials in J, together with zero, is an α-Poisson ideal since $J' = \bigcup_n J_n$.

(iii) The ideal $IA[x; \alpha, \delta]_p$ is a Poisson ideal of $A[x; \alpha, \delta]_p$ since $\{a, x\} = \alpha(a)x + \delta(a)$ for all $a \in I$. Moreover, there exists a Poisson isomorphism from $A[x; \alpha, \delta]_p / IA[x; \alpha, \delta]_p$ onto $(A/I)[x; \bar{\alpha}, \bar{\delta}]_p$ defined by

$$(a_0 + \cdots + a_n x^n) + IA[x; \alpha, \delta]_p \mapsto (a_0 + I) + \cdots + (a_n + I)x^n.$$

(iv) It is trivial because of $\{x, y\} = axy$ in $A[x; \alpha]_p[y; \beta]_p$. □

3.3 Lemma. *Let $B = A[x, x^{-1}; \alpha]_p$. Then B is Poisson simple if and only if:*

(i) *A is α-Poisson simple;*
(ii) *For each positive integer n, $n\alpha$ is not inner.*

Proof. Suppose that B is Poisson simple. If I is an α-Poisson ideal of A, then IB is a Poisson ideal of B by 3.2(iii). If $n\alpha$ is inner determined by an invertible element $a \in A$, then $a^{-1}x^n - 1$ is a Poisson central element of B. Thus $(a^{-1}x^n - 1)B$ is a nontrivial Poisson ideal of B.

Conversely, let J be a nonzero Poisson ideal of B and let $C = A[x; \alpha]_p \subseteq B$. Then $J \cap C$ is a nonzero Poisson ideal of C. Choose a nonzero element $f \in J \cap C$ such that f has the minimal degree n among nonzero elements in $J \cap C$. Since $(J \cap C)_n$ is a nonzero α-Poisson ideal of A by 3.2(ii), we may assume that the leading coefficient of f is unity. Set

$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and suppose that there exists $a_i \neq 0$ for some i. Since $\{f, x\}x^{-1}$ and $\{r, f\} - n\alpha(r)f$ are elements of $J \cap C$ with degree less than n, a_iA is a nonzero α-Poisson ideal and

$$(n - i)a_i\alpha(r) = \{r, a_i\}$$

for all $r \in A$, and thus a_i is invertible and $(n - i)\alpha$ is inner, a contradiction. It follows that $J = B$ since $f = x^n \in J$ is an invertible element of B. □

3.4 Proposition. *Let $B = A[x; \alpha]_p$.*

(i) *The prime Poisson ideals of B containing x are precisely the ideals of the form $I + xB$, where I is a prime Poisson ideal of A.*
(ii) *If I is an α-prime Poisson ideal of A then IB is a prime Poisson ideal of B.*
(iii) *If P is a prime Poisson ideal not containing x then $P \cap A$ is an α-prime Poisson ideal of A.*
(iv) *If $n\alpha$ is inner for some positive integer n, then there exists a Poisson central element $y \in B$ such that y is transcendental over A and B is a finitely generated $A[y]$-module.*
(v) *If A is α-Poisson simple and there is no positive integer n such that $n\alpha$ is inner then every prime Poisson ideal of B contains x.*

Proof. (i) Since x is a Poisson normal element, xB is a Poisson ideal. Hence the result follows from the fact that A is Poisson isomorphic to B/xB.

(ii) By 3.2(iii), IB is a Poisson ideal of B since I is α-Poisson and there is a Poisson isomorphism from B/IB onto $(A/I)[x; \bar{\alpha}]_p$. Hence IB is a prime Poisson ideal of B.

(iii) By 3.2(ii), $P \cap A$ is a prime Poisson ideal of A. Moreover, $P \cap A$ is α-stable since $x \notin P$ and $\alpha(a)x = \{a, x\} \in P$ for all $a \in P \cap A$.

(iv) If $n\alpha$ is an inner Poisson derivation determined by an invertible element $a \in A$ then $y = a^{-1}x^n - 1$ is a Poisson central element of B and thus B is a finitely generated $A[y]$-module since x is integral over $A[y]$.

(v) It follows immediately from 3.3. □

3.5. We recall Oh (1999, Theorem 2.4) and Brown and Goodearl (2002, Chapter II.8): A Poisson **k**-algebra A is said to satisfy the Dixmier-Moeglin equivalence if the following conditions are equivalent (for all prime Poisson ideals P of A):

(i) P is Poisson primitive (i.e., there exists a maximal ideal M of A such that P is the largest Poisson ideal contained in M).
(ii) P is rational (i.e., the Poisson center of the quotient field of A/P is algebraic over **k**).
(iii) P is locally closed (i.e., the intersection of all prime Poisson ideals properly containing P is strictly larger than P).

Theorem. *The Poisson algebra $A_2 = A_{2,\Gamma}^{P,Q}$ given in* 2.7 *satisfies the Dixmier-Moeglin equivalence.*

Proof. Since A_2 is finitely generated, we have the implications

$$\text{locally closed} \Rightarrow \text{Poisson primitive} \Rightarrow \text{rational}$$

by Oh (1999, 1.7 and 1.10), and thus it is enough to prove that every rational prime Poisson ideal is locally closed.

Let $\operatorname{pspec} A_2$ be the set of all prime Poisson ideals of A_2 and set

$$X = \{P \in \operatorname{pspec} A_2 \mid y_1 \in P\}$$
$$Y = \{P \in \operatorname{pspec} A_2 \mid x_1 \in P\}$$
$$Z = \{P \in \operatorname{pspec} A_2 \mid y_1 \notin P, x_1 \notin P\}.$$

Since

$$A_2/y_1A_2 \cong \mathbf{k}[x_1][y_2; \alpha]_p[x_2; \beta]_p = B$$
$$A_2/x_1A_2 \cong \mathbf{k}[y_1][y_2; \alpha']_p[x_2; \beta']_p = C,$$

where

$$\alpha = (p_2 - \gamma_{12})x_1\frac{\partial}{\partial x_1}, \qquad \beta = (q_1 - p_2 + \gamma_{12})x_1\frac{\partial}{\partial x_1} - q_2y_2\frac{\partial}{\partial y_2}$$

$$\alpha' = \gamma_{12}y_1\frac{\partial}{\partial y_1}, \qquad \beta' = -(q_1 + \gamma_{12})y_1\frac{\partial}{\partial y_1} - q_2y_2\frac{\partial}{\partial y_2},$$

X and Y are homeomorphic to pspec B and pspec C. Set

$$z = (q_2 - p_2)y_2x_2 + (q_1 - p_1)y_1x_1 \in A_2.$$

Then z is a Poisson normal element, and a prime Poisson ideal of A_2 containing either y_2 or x_2 contains either y_1 or x_1 since

$$\{x_2, y_2\} = q_2y_2x_2 + (q_1 - p_1)y_1x_1.$$

Moreover, since

$$A_2[y_1^{-1}, x_1^{-1}, y_2^{-1}] = \mathbf{k}[y_1^{\pm 1}][x_1^{\pm 1}; \alpha]_p[y_2^{\pm 1}; \beta]_p[z; \gamma]_p = D,$$

where

$$\alpha = -q_1y_1\frac{\partial}{\partial y_1}$$

$$\beta = \gamma_{12}y_1\frac{\partial}{\partial y_1} + (p_2 - \gamma_{12})x_1\frac{\partial}{\partial x_1}$$

$$\gamma = -q_1y_1\frac{\partial}{\partial y_1} + q_1x_1\frac{\partial}{\partial x_1} - q_2y_2\frac{\partial}{\partial y_2},$$

Z is homeomorphic to a subspace of pspec D.

Let a prime Poisson ideal P of A_2 be rational and let X' (respectively Y', Z') be the prime Poisson ideals of X (respectively, Y, Z) properly containing P. Then $\cap X'$ (respectively $\cap Y'$, $\cap Z'$) contains an element $a_1 \notin P$ (respectively $a_2 \notin P$, $a_3 \notin P$) since B, C, D satisfy the Dixmier-Moeglin equivalence by Oh (1999, §2). It follows that the intersection $(\cap X') \cap (\cap Y') \cap (\cap Z')$ of all prime Poisson ideals properly containing P is strictly larger than P since

$$a_1a_2a_3 \in (\cap X') \cap (\cap Y') \cap (\cap Z'), \qquad a_1a_2a_3 \notin P.$$

It completes the proof. □

3.6. In 2.7, the coordinate ring A_2 of Poisson symplectic 4-space is the Poisson algebra $\mathbf{k}[y_1, x_1, y_2, x_2]$ with Poisson bracket

$$\{y_1, y_2\} = y_1y_2 \qquad \{x_1, x_2\} = -x_1x_2$$

$$\{y_1, x_2\} = y_1x_2 \qquad \{x_1, y_2\} = -x_1y_2$$

$$\{y_1, x_1\} = 2y_1x_1 \qquad \{y_2, x_2\} = 2y_2x_2 + 2y_1x_1.$$

Here we find the prime Poisson ideals of A_2 that give a motivation for this article. Note that y_1 and x_1 are Poisson normal elements of A_2.

(i) The prime Poisson ideals containing x_1: Observe that

$$A_2/x_1A_2 \cong \mathbf{k}[y_1][y_2; \alpha]_p[x_2; \beta]_p, \tag{3.1}$$

where $\alpha = y_1\frac{\partial}{\partial y_1}$, $\beta = y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}$. Set

$$A = \mathbf{k}[y_1][y_2; \alpha]_p, \qquad B = \mathbf{k}[y_1][y_2; \alpha]_p[x_2; \beta]_p = A[x_2; \beta]_p.$$

All prime Poisson ideals of A containing y_2 are of the form $I + y_2A$ by 3.4(i), where I is a prime ideal of $\mathbf{k}[y_1]$. Note that $A = \mathbf{k}[y_2][y_1; \alpha']_p$ by 3.2(iv), where $\alpha' = -y_2\frac{\partial}{\partial y_2}$, and that α' is extended to A, still denoted by α', by setting $\alpha'(y_1) = 0$. Let P be a nonzero prime Poisson ideal of A not containing y_2 and let $f = f_0 + f_1y_2 + \cdots + f_ry_2^r$, $f_i \in \mathbf{k}[y_1]$, be a nonzero element of P. Since $\alpha'(P)y_2 = \{P, y_2\} \subseteq P$, P is α'-stable and each $f_iy_2^r$ is an eigenvector of α' with eigenvalue $-r$, and thus each $f_i \in P$. It follows that P contains a nonzero element $g = g_0 + g_1y_1 + \cdots + g_ny_1^n \in \mathbf{k}[y_1]$. Since $P \cap \mathbf{k}[y_1]$ is α-stable, we have that $y_1 \in P$ by applying α on g. Therefore all prime Poisson ideals of A are

$$0, \qquad I + y_2A, \qquad J + y_1A,$$

where I and J are prime ideals of $\mathbf{k}[y_1]$ and $\mathbf{k}[y_2]$, respectively.

Extend α and β to B in order that, for each $f \in B$, $\alpha(f)y_2 = \{f, y_2\}$ and $\beta(f)x_2 = \{f, x_2\}$ respectively. Moreover there is a Poisson derivation γ on B such that $\gamma(f)y_1 = \{f, y_1\}$ for all $f \in B$ by 1.1. Since every prime Poisson ideal of B not containing y_1, y_2, x_2 is (α, β, γ)-stable and each monomial $y_1^ry_2^sx_2^t \in B$ is a common eigenvector of α, β, γ with eigenvalue $r - 2t$, $r + 2s$, $-s - t$, respectively, every nonzero prime Poisson ideal of B contains one of y_1, y_2, x_2. Therefore all prime Poisson ideals of A_2 containing x_1 are

$$\begin{array}{ll}
x_1A_2, & x_1A_2 + x_2A_2 \\
x_1A_2 + y_2A_2, & y_1A_2 + x_1A_2 \\
JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & IA_2 + x_1A_2 + y_2A_2 + x_2A_2 \\
KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & IA_2 + x_1A_2 + y_2A_2 + x_2A_2 \\
JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & KA_2 + y_1A_2 + x_1A_2 + y_2A_2
\end{array}$$

where I, J, K are prime ideals of $\mathbf{k}[y_1], \mathbf{k}[y_2], \mathbf{k}[x_2]$, respectively.

(ii) The prime Poisson ideals containing y_1: Observe that

$$A_2/y_1A_2 \cong \mathbf{k}[x_1][y_2; \alpha]_p[x_2; \beta]_p,$$

where $\alpha = -x_1\frac{\partial}{\partial x_1}$, $\beta = -x_1\frac{\partial}{\partial x_1} + 2y_2\frac{\partial}{\partial y_2}$. Replacing x_1 in $\mathbf{k}[x_1][y_2; \alpha]_p[x_2; \beta]_p$ by y_1 in the Poisson algebra given in the right hand of (3.1), all prime Poisson ideals of A_2

containing y_1 are

$$\begin{array}{ll} y_1A_2, & y_1A_2 + x_2A_2 \\ y_1A_2 + y_2A_2, & y_1A_2 + x_1A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2 \\ KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & KA_2 + y_1A_2 + x_1A_2 + y_2A_2 \end{array}$$

where L, J, K are prime ideals of $\mathbf{k}[x_1], \mathbf{k}[y_2], \mathbf{k}[x_2]$, respectively.

(iii) The prime Poisson ideals containing neither y_1 nor x_1: If a prime Poisson ideal P containing either $y_2 \in P$ or $x_2 \in P$ then $y_1 \in P$ or $x_1 \in P$ since $\{y_2, x_2\} = 2y_2x_2 + 2y_1x_1$. Hence we may assume that $y_2 \notin P$ and $x_2 \notin P$.

Set $z = y_2x_2 + y_1x_1$. Then z is a Poisson normal element of A_2 and

$$A_2[y_1^{-1}, x_1^{-1}, y_2^{-1}] = \mathbf{k}[y_1^{\pm1}][x_1^{\pm1}; \alpha]_p[y_2^{\pm1}; \beta]_p[z; \gamma]_p,$$

where

$$\alpha = 2y_1\frac{\partial}{\partial y_1}$$
$$\beta = y_1\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_1}$$
$$\gamma = 2y_1\frac{\partial}{\partial y_1} - 2x_1\frac{\partial}{\partial x_1} + 2y_2\frac{\partial}{\partial y_2}.$$

Set $A = \mathbf{k}[y_1^{\pm1}][x_1^{\pm1}; \alpha]_p[y_2^{\pm1}; \beta]_p$, $B = A[z; \gamma]_p$. It is proven as in the third paragraph of the case (i) that A has no nontrivial prime Poisson ideal. Suppose that there exists a nontrivial Poisson ideal I of A and let P be a prime ideal minimal over I. Then the largest $\mathscr{H}$-stable ideal $(P : \mathscr{H})$ contained in P is a prime Poisson ideal containing I by Dixmier (1996, 3.3.2), where $\mathscr{H}$ is the set of all hamiltonians in A. Hence A is Poisson simple, in particular, γ-Poisson simple. Suppose that $n\gamma$ is inner for some positive integer n. Then there exists an invertible element $f \in A$ such that $n\gamma(h) = f^{-1}\{h, f\}$ for all $h \in A$. Hence $f = ay_1^rx_1^sy_2^t$ for some $a \in \mathbf{k}^\times$ and integers r, s, t and we get $n = 0$ by calculating $n\gamma(y_1)$, $n\gamma(x_1)$, and $n\gamma(y_2)$, a contradiction. Hence $n\gamma$ is not inner for each positive integer n. It follows that every nonzero prime Poisson ideal of B contains z by 3.4(v), and thus the prime Poisson ideals of B are only zero and zB.

Therefore, by (i), (ii), and (iii), all prime Poisson ideals of A_2 are as follows:

$$\begin{array}{ll} 0, & zA_2 \\ x_1A_2, & x_1A_2 + x_2A_2 \\ x_1A_2 + y_2A_2, & y_1A_2 + x_1A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & IA_2 + x_1A_2 + y_2A_2 + x_2A_2 \\ KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & IA_2 + x_1A_2 + y_2A_2 + x_2A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & KA_2 + y_1A_2 + x_1A_2 + y_2A_2 \end{array}$$

$$\begin{array}{ll} y_1A_2, & y_1A_2 + x_2A_2 \\ y_1A_2 + y_2A_2, & y_1A_2 + x_1A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2 \\ KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2 \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & KA_2 + y_1A_2 + x_1A_2 + y_2A_2, \end{array}$$

where $z = y_2x_2 + y_1x_1$ and I, J, K, L are prime ideals of $\mathbf{k}[y_1]$, $\mathbf{k}[y_2], \mathbf{k}[x_2]$, $\mathbf{k}[x_1]$, respectively.

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