A CONSTRUCTION OF AN ITERATED ORE EXTENSION

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ABSTRACT. Let B be a Poisson algebra $\mathbb{C}[x_1,\dots,x_k]$ with Poisson bracket such that

$$\{x_j,x_i\}=c_{ji}x_ix_j+p_{ji}$$

for all $j>i$, where $c_{ji}\in\mathbb{C}$ and $p_{ji}\in\mathbb{C}[x_1,\dots,x_i]$. Here we obtain an iterated skew polynomial algebra such that its semiclassical limit is equal to B and the results are illustrated by examples.

1. Introduction

Recall the star product in [10, 1.1]. Let $R=(R,\{-,-\})$ be a Poisson algebra and let Q be a quantization of R with a star product $*$. Then Q is a $\mathbb{C}[[\hbar]]$-algebra $R[[\hbar]]$ such that for any $a,b\in R\subset Q=R[[\hbar]]$,

$$a*b=ab+B_1(a,b)\hbar+B_2(a,b)\hbar^2+\dots$$

subject to

$$\{a,b\}=\hbar^{-1}(a*b-b*a)|_{\hbar=0}, \tag{1.1}$$

where $B_i:R\times R\longrightarrow R$ are bilinear products. In general, the star product is as follows: for any $f=\sum_{n\ge0}f_n\hbar^n, g=\sum_{n\ge0}g_n\hbar^n\in Q$

$$(\sum_{n\ge0}f_n\hbar^n)*(\sum_{n\ge0}g_n\hbar^n)=\sum_{k,l\ge0}f_kg_l\hbar^{k+l}+\sum_{k,l\ge0,m\ge1}B_m(f_k,g_l)\hbar^{k+l+m}.$$

It is well-known that we can recover the Poisson algebra $R=Q/\hbar Q$ with Poisson bracket (1.1) from Q since $\hbar$ is a nonzero, nonunit, non-zero-divisor and central element such that $Q/\hbar Q$ is commutative. But it seems that the star product in Q is complicate and Q is difficult to understand at an algebraic point of view since it is too big. For instance, if λ is a nonzero element of $\mathbb{C}$ then $\hbar-\lambda$ is a unit in Q and thus $Q/(\hbar-\lambda)Q$ is trivial. Hence it seems that we need an appropriate $\mathbb{F}$-subalgebra A of Q such that A contains all generators of Q, $\hbar\in A$ and A is understandable at an algebraic point of view, where $\mathbb{F}$ is a subring of $\mathbb{C}[[\hbar]]$.

Suppose that A is an algebra and let $\hbar\in A$ be a nonzero, nonunit, non-zero-divisor and central element such that $A/\hbar A$ is commutative. Then $A/\hbar A$ is a nontrivial commutative algebra as well as a Poisson algebra with the Poisson bracket

$$\{\overline{a},\overline{b}\}=\overline{\hbar^{-1}(ab-ba)} \tag{1.2}$$

for $\overline{a},\overline{b}\in A/\hbar A$. Note that (1.1) is equal to (1.2). Further, if there is an element $0\ne\lambda\in\mathbb{C}$ such that $\hbar-\lambda$ is a nonunit in A then we obtain a nontrivial algebra $A/(\hbar-\lambda)A$ with the multiplication induced by that of A. The Poisson algebra $A/\hbar A$ is called a *semiclassical limit* of

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A and the nontrivial algebra $A/(\hbar-\lambda)A$ is called a *deformation* of A or $A/\hbar A$ in [5, 2.1]. The element $\hbar\in A$ inducing the Poisson algebra $A/\hbar A$ is called a *regular element* of A. Namely, by a regular element $\hbar\in A$ we mean a nonzero, nonunit, non-zero-divisor and central element of A such that $A/\hbar A$ is commutative. (An anonymous referee suggested to use the terminology 'regular element' while several papers for semiclassical limits were written even though there are many concepts for 'regular element' as in [12] and [6]. We hope that a nice terminology for this concept is given.) In general, let A be an $\mathbb{F}$-algebra generated by $x_1,\dots,x_n$ with relations $f_1,\dots,f_r$ and let $\lambda\in\mathbb{C}$, where $\mathbb{F}$ is a subring of $\mathbb{C}[[\hbar]]$ containing $\mathbb{C}[\hbar]$ and f_i are elements of the free $\mathbb{F}$-algebra on the set $\{x_1,\dots,x_n\}$. Assume that $g|_{\hbar=\lambda}$, $f_i|_{\hbar=\lambda}$ make sense mathematically for all $g\in A$ and $i=1,\dots,r$. Denote by A_λ the $\mathbb{C}$-algebra generated by $x_1,\dots,x_n$ with the relations $f_1|_{\hbar=\lambda},\dots,f_r|_{\hbar=\lambda}$ and let φ be the evaluation map from A onto A_λ defined by $g\mapsto g|_{\hbar=\lambda}$. Then φ is a $\mathbb{C}$-algebra epimorphism and $A/\ker\varphi\cong A_\lambda$. In particular, if $\ker\varphi\neq A$ then A_λ is nontrivial and the multiplication of A_λ is induced by that of A. We still call the nontrivial algebra A_λ a *deformation* of $A/\hbar A$.

Let B_k be a Poisson $\mathbb{C}$-algebra $\mathbb{C}[x_1,\dots,x_k]$ with Poisson bracket such that for all $j>i$,

$$\{x_j,x_i\}=c_{ji}x_ix_j+p_{ji}, \tag{1.3}$$

where $c_{ji}\in\mathbb{C}$ and $p_{ji}\in\mathbb{C}[x_1,\dots,x_i]$. A main aim of the article is to give how to construct an $\mathbb{F}$-algebra which is presented by an iterated skew polynomial algebra such that its semiclassical limit is equal to the given Poisson algebra B_k.

Let t be an indeterminate and let $\mathbb{C}[[t-1]]$ be the ring of formal power series over $\mathbb{C}$ at $t-1$. Namely,

$$\mathbb{C}[[t-1]]=\left\{\sum_{i=0}^{\infty}b_i(t-1)^i\mid b_i\in\mathbb{C}\right\}.$$

Note that $\mathbb{C}[[t-1]]$ is an integral domain, that $\mathbb{C}[t]\subseteq\mathbb{C}[[t-1]]$ and that a nonzero element $\sum_{i=0}^{\infty}b_i(t-1)^i$ is a unit in $\mathbb{C}[[t-1]]$ if and only if $b_0\neq0$. We assume throughout the article that $\mathbb{F}$ is a subring of $\mathbb{C}[[t-1]]$ containing $\mathbb{C}[t]$, namely

$$\mathbb{C}[t]\subseteq\mathbb{F}\subseteq\mathbb{C}[[t-1]].$$

Let

$$A_{k-1}=\mathbb{F}[x_1][x_2;\beta_2,\nu_2]\dots[x_{k-1};\beta_{k-1},\nu_{k-1}]$$

be an iterated skew polynomial $\mathbb{F}$-algebra and let β_k,ν_k be $\mathbb{F}$-linear maps from A_{k-1} into itself. In this article, we find necessary and sufficient conditions for β_k and ν_k such that there exists a skew polynomial algebra $A_k=A_{k-1}[x_k;\beta_k,\nu_k]$ under suitable conditions. (See Lemma 2.2 and Theorem 2.4.) Hence, using induction on k repeatedly, we can get iterated skew polynomial algebras from the result. Next we observe that $t-1$ is a regular element of A_k and find a condition such that the Poisson algebra $B_k=\mathbb{C}[x_1,\dots,x_k]$ with Poisson bracket (1.3) is Poisson isomorphic to the semiclassical limit $A_k/(t-1)A_k$. (See Corollary 2.8 and [3, §2].) Finally we give examples illustrating the results.

Recall several basic terminologies. (1) Given an $\mathbb{F}$-endomorphism β on an $\mathbb{F}$-algebra R, an $\mathbb{F}$-linear map ν is said to be a *left β-derivation* on R if $\nu(ab)=\beta(a)\nu(b)+\nu(a)b$ for all $a,b\in R$. For such a pair (β,ν), we denote by $R[z;\beta,\nu]$ the skew polynomial $\mathbb{F}$-algebra. Refer to [6, Chapter 2] for details of a skew polynomial algebra.

(2) A commutative $\mathbb{C}$-algebra R is said to be a *Poisson algebra* if there exists a bilinear product $\{-,-\}$ on R, called a *Poisson bracket*, such that $(R,\{-,-\})$ is a Lie algebra with $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all $a,b,c \in R$. A derivation α on R is said to be a *Poisson derivation* if $\alpha(\{a,b\}) = \{\alpha(a),b\} + \{a,\alpha(b)\}$ for all $a,b \in R$. Let α be a Poisson derivation on R and let δ be a derivation on R such that

$$\delta(\{a,b\}) - \{\delta(a),b\} - \{a,\delta(b)\} = \alpha(a)\delta(b) - \delta(a)\alpha(b) \tag{1.4}$$

for all $a,b \in R$. By [15, 1.1], the commutative polynomial $\mathbb{C}$-algebra $R[z]$ is a Poisson algebra with Poisson bracket $\{z,a\} = \alpha(a)z + \delta(a)$ for all $a \in R$. Such a Poisson polynomial algebra $R[z]$ is denoted by $R[z;\alpha,\delta]_p$ in order to distinguish it from skew polynomial algebras. If $\alpha = 0$ then we write $R[z;\delta]_p$ for $R[z;0,\delta]_p$ and if $\delta = 0$ then we write $R[z;\alpha]_p$ for $R[z;\alpha,0]_p$.

2. A construction of an iterated skew polynomial algebra

Set $A_1 = \mathbb{F}[x_1]$ and let A_n, $n > 1$, be an iterated skew polynomial $\mathbb{F}$-algebra

$$A_n = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\dots[x_n;\beta_n,\nu_n].$$

By monomials in A_n we mean finite products of x_i's together with the unity 1. A monomial X is said to be *standard* if X is of the form

$$X = 1 \text{ or } X = x_{i_1}x_{i_2}\cdots x_{i_k} \qquad (1 \le i_1 \le i_2 \le \ldots \le i_k \le n).$$

Note that the set of all standard monomials of A_n forms an $\mathbb{F}$-basis.

Let β and ν be $\mathbb{F}$-linear maps from an $\mathbb{F}$-algebra R into itself. The following lemma is well known, e.g. see [7, p.177].

Lemma 2.1. *The following conditions are equivalent:*
(1) The $\mathbb{F}$-linear map $\phi : R \to M_2(R)$ by

$$\phi(r) = \begin{pmatrix} \beta(r) & \nu(r) \\ 0 & r \end{pmatrix}$$

for all $r \in R$, is an $\mathbb{F}$-algebra homomorphism
(2) β and ν are an endomorphism and a left β-derivation on R respectively.

In an iterated skew polynomial $\mathbb{F}$-algebra

$$A_{k-1} = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\dots[x_{k-1};\beta_{k-1},\nu_{k-1}],$$

assume that β_j, ν_j $(j = 2,\dots,k-1)$ satisfy

$$\beta_j(x_i) = a_{ji}x_i,\ a_{ji} \in \mathbb{F} \qquad (1 \le i < j < k) \tag{2.1}$$

$$\nu_j(x_i) = u_{ji} \in A_i \qquad (1 \le i < j < k). \tag{2.2}$$

We are going to construct a skew polynomial $\mathbb{F}$-algebra

$$A_k = A_{k-1}[x_k;\beta_k,\nu_k] = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\dots[x_k;\beta_k,\nu_k]$$

such that β_k, ν_k satisfy the following conditions

$$\beta_k(1) = 1,\ \beta_k(x_i) = a_{ki}x_i,\ a_{ki} \in \mathbb{F} \qquad (1 \le i \le k-1), \tag{2.3}$$

$$\nu_k(1) = 0,\ \nu_k(x_i) = u_{ki} \in A_i \qquad (1 \le i \le k-1). \tag{2.4}$$

The following statement gives us necessary conditions for the existence of the skew polynomial $\mathbb{F}$-algebra $A_k = A_{k-1}[x_k;\beta_k,\nu_k]$ over A_{k-1}.

Lemma 2.2. *If there exists a skew polynomial $\mathbb{F}$-algebra $A_k = A_{k-1}[x_k; \beta_k, \nu_k]$ such that β_k, ν_k are subject to (2.3), (2.4) then β_k, ν_k satisfy the following conditions*

$$\beta_k(u_{ji}) = a_{kj}a_{ki}u_{ji} \qquad (1 \le i < j < k), \tag{2.5}$$

$$a_{kj}x_ju_{ki} + u_{kj}x_i = a_{ji}u_{ki}x_j + a_{ki}a_{ji}x_iu_{kj} + \nu_k(u_{ji}) \quad (1 \le i < j < k). \tag{2.6}$$

Proof. Let $1 \le i < j \le k-1$. Since β_k is an $\mathbb{F}$-algebra endomorphism, we have that

$$\beta_k(x_jx_i) = \beta_k(\beta_j(x_i)x_j + \nu_j(x_i)) = a_{kj}a_{ki}a_{ji}x_ix_j + \beta_k(u_{ji})$$

and

$$\begin{aligned}\beta_k(x_jx_i) &= \beta_k(x_j)\beta_k(x_i) = a_{kj}a_{ki}x_jx_i \\ &= a_{kj}a_{ki}(\beta_j(x_i)x_j + \nu_j(x_i)) = a_{kj}a_{ki}a_{ji}x_ix_j + a_{kj}a_{ki}u_{ji}\end{aligned}$$

by (2.1)-(2.4). Hence we get (2.5).

Similarly, since ν_k is a left β_k-derivation, we have that

$$\nu_k(x_jx_i) = \nu_k(x_j)x_i + \beta_k(x_j)\nu_k(x_i) = u_{kj}x_i + a_{kj}x_ju_{ki}$$

and

$$\begin{aligned}\nu_k(x_jx_i) &= \nu_k(\beta_j(x_i)x_j + \nu_j(x_i)) = \nu_k(a_{ji}x_ix_j + u_{ji}) \\ &= a_{ji}(\nu_k(x_i)x_j + \beta_k(x_i)\nu_k(x_j)) + \nu_k(u_{ji}) \\ &= a_{ji}u_{ki}x_j + a_{ki}a_{ji}x_iu_{kj} + \nu_k(u_{ji})\end{aligned}$$

by (2.1)-(2.4). Hence we get (2.6). □

Lemma 2.3. *For $1 \le i < j \le k-1$, let all $\beta_j, \nu_j, a_{ji}, u_{ji}$ satisfy (2.1), (2.2). Let β_k, ν_k be $\mathbb{F}$-linear maps from A_{k-1} into itself subject to the conditions (2.3) and (2.4). If β_k and ν_k satisfy (2.5) and (2.6) then the following conditions hold.*

$$\beta_k(x_j)\beta_k(x_i) = \beta_k\beta_j(x_i)\beta_k(x_j) + \beta_k\nu_j(x_i), \tag{2.7}$$

$$\beta_k(x_j)\nu_k(x_i) + \nu_k(x_j)x_i = \beta_k\beta_j(x_i)\nu_k(x_j) + \nu_k\beta_j(x_i)x_j + \nu_k\nu_j(x_i) \tag{2.8}$$

Proof. Since A_{k-1} is an iterated skew polynomial $\mathbb{F}$-algebra, the equations (2.7) and (2.8) follow from (2.5) and (2.6), respectively, by (2.1)-(2.4). □

In the following theorem, we see that (2.5) and (2.6) are sufficient conditions for the existence of the skew polynomial $\mathbb{F}$-algebra $A_k = A_{k-1}[x_k; \beta_k, \nu_k]$ over A_{k-1}.

Theorem 2.4. *For $1 \le i < j \le k-1$, let all $\beta_j, \nu_j, a_{ji}, u_{ji}$ satisfy (2.1), (2.2). Given $\mathbb{F}$-linear maps β_k, ν_k from A_{k-1} into itself subject to (2.3), (2.4), if β_k and ν_k satisfy the conditions (2.5), (2.6) then there exists an iterated skew polynomial $\mathbb{F}$-algebra*

$$A_k = A_{k-1}[x_k; \beta_k, \nu_k] = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \ldots [x_k; \beta_k, \nu_k].$$

Proof. It is enough to show that there exist an $\mathbb{F}$-algebra endomorphism β_k on A_{k-1} and a left β_k-derivation ν_k subject to the conditions (2.3) and (2.4). Note that the set of all standard monomials forms an $\mathbb{F}$-basis of A_{k-1}. For any standard monomials $x_{i_1} \cdots x_{i_r} \in A_{k-1}$, define $\mathbb{F}$-linear maps β_k and ν_k from A_{k-1} into itself by

$$\beta_k(1) = 1, \quad \beta_k(x_{i_1} \cdots x_{i_r}) = (a_{ki_1}x_{i_1}) \cdots (a_{ki_r}x_{i_r}), \tag{2.9}$$

$$\nu_k(1) = 0, \quad \nu_k(x_{i_1} \cdots x_{i_r}) = \sum_{\ell=1}^{r} (a_{ki_1}x_{i_1}) \cdots (a_{ki_{\ell-1}}x_{i_{\ell-1}})u_{ki_\ell}(x_{i_{\ell+1}} \cdots x_{i_r}). \tag{2.10}$$

Observe that these $\mathbb{F}$-linear maps β_k and ν_k satisfy (2.3) and (2.4). We will show that the map β_k defined by (2.9) is an $\mathbb{F}$-algebra endomorphism and the map ν_k defined by (2.10) is a left β_k-derivation by using Lemma 2.1.

Let $\mathbb{F}\langle S_{k-1}\rangle$ be the free $\mathbb{F}$-algebra on the set $S_{k-1} = \{x_1, \dots, x_{k-1}\}$. Define an $\mathbb{F}$-algebra homomorphism $f : \mathbb{F}\langle S_{k-1}\rangle \to M_2(A_{k-1})$ by

$$f(x_i) = \begin{pmatrix} \beta_k(x_i) & \nu_k(x_i) \\ 0 & x_i \end{pmatrix} \qquad (1 \le i < k).$$

Let us show that

$$f(\nu_j(x_i)) = \begin{pmatrix} \beta_k\nu_j(x_i) & \nu_k\nu_j(x_i) \\ 0 & \nu_j(x_i) \end{pmatrix} \tag{2.11}$$

for $1 \le i < j < k$. For any standard monomial $X = x_{i_1} \cdots x_{i_r}$ in A_{k-1}, by (2.9) and (2.10),

$$\begin{aligned}
\nu_k(X) &= \sum_{\ell=1}^{r} \beta_k(x_{i_1} \cdots x_{i_{\ell-1}})\nu_k(x_{i_\ell})(x_{i_{\ell+1}} \cdots x_{i_r}) \\
&= \sum_{\ell=1}^{r-1} \beta_k(x_{i_1} \cdots x_{i_{\ell-1}})\nu_k(x_{i_\ell})(x_{i_{\ell+1}} \cdots x_{i_r}) + \beta_k(x_{i_1} \cdots x_{i_{r-1}})\nu_k(x_{i_r}) \\
&= \nu_k(x_{i_1} \cdots x_{i_{r-1}})x_{i_r} + \beta_k(x_{i_1} \cdots x_{i_{r-1}})\nu_k(x_{i_r}).
\end{aligned}$$

In particular, if Xx_j is standard (thus $i_r \le j$) then

$$\nu_k(Xx_j) = \beta_k(X)\nu_k(x_j) + \nu_k(X)x_j. \tag{2.12}$$

Let us verify first that

$$f(X) = \begin{pmatrix} \beta_k(X) & \nu_k(X) \\ 0 & X \end{pmatrix} \tag{2.13}$$

for any standard monomial $X = x_{i_1} \cdots x_{i_r}$ in A_{k-1} of length r. We proceed by induction on r. If $r = 1$ then (2.13) is true trivially. Assume that $r > 1$ and that (2.13) holds for any standard monomial of length $< r$. Set $Y = x_{i_1} \cdots x_{i_{r-1}}$. Then Y is a standard monomial of length $r - 1$ and $X = Yx_{i_r}$. Thus (2.13) holds as follows:

$$\begin{aligned}
f(X) &= f(Yx_{i_r}) = f(Y)f(x_{i_r}) \\
&= \begin{pmatrix} \beta_k(Y) & \nu_k(Y) \\ 0 & Y \end{pmatrix} \begin{pmatrix} \beta_k(x_{i_r}) & \nu_k(x_{i_r}) \\ 0 & x_{i_r} \end{pmatrix} && \text{(by induction hypothesis)} \\
&= \begin{pmatrix} \beta_k(Y)\beta_k(x_{i_r}) & \beta_k(Y)\nu_k(x_{i_r}) + \nu_k(Y)x_{i_r} \\ 0 & Yx_{i_r} \end{pmatrix} \\
&= \begin{pmatrix} \beta_k(X) & \nu_k(X) \\ 0 & X \end{pmatrix}. && \text{(by (2.9), (2.12))}
\end{aligned}$$

Let $\nu_j(x_i) = \sum_\ell b_\ell X_\ell$, where all $b_\ell \in \mathbb{F}$ and X_ℓ are standard monomials of A_i. Since f is an $\mathbb{F}$-algebra homomorphism, we have

$$\begin{aligned} f(\nu_j(x_i)) &= \sum_\ell b_\ell f(X_\ell) \\ &= \sum_\ell b_\ell \begin{pmatrix} \beta_k(X_\ell) & \nu_k(X_\ell) \\ 0 & X_\ell \end{pmatrix} \qquad \text{(by (2.13))} \\ &= \begin{pmatrix} \beta_k(\sum_\ell b_\ell X_\ell) & \nu_k(\sum_\ell b_\ell X_\ell) \\ 0 & \sum_\ell b_\ell X_\ell \end{pmatrix} \\ &= \begin{pmatrix} \beta_k \nu_j(x_i) & \nu_k \nu_j(x_i) \\ 0 & \nu_j(x_i) \end{pmatrix}. \end{aligned}$$

Thus (2.11) holds.

Note that A_{k-1} is an $\mathbb{F}$-algebra generated by $x_1, \ldots, x_{k-1}$ with relations

$$x_j x_i - \beta_j(x_i) x_j - \nu_j(x_i) \qquad (1 \le i < j < k).$$

Namely, A_{k-1} is isomorphic to the $\mathbb{F}$-algebra $\mathbb{F}\langle S_{k-1} \rangle / I$, where I is the ideal generated by

$$x_j x_i - \beta_j(x_i) x_j - \nu_j(x_i) \quad (1 \le i < j < k).$$

Since f is an $\mathbb{F}$-algebra homomorphism, it is easy to check that $I \subseteq \ker f$ by (2.7), (2.8) and (2.11). Hence there exists an $\mathbb{F}$-algebra homomorphism $\phi : A_{k-1} \to M_2(A_{k-1})$ such that

$$\phi(x_i) = \begin{pmatrix} \beta_k(x_i) & \nu_k(x_i) \\ 0 & x_i \end{pmatrix}$$

for $1 \le i < k$. By Lemma 2.1, β_k is an $\mathbb{F}$-algebra endomorphism on A_{k-1} and ν_k is a left β_k-derivation on A_{k-1} as claimed. □

Remark 2.5. Retain the notations of Theorem 2.4.
(1) If $a_{ki} \neq 0$ for all $1 \le i < k$ then β_k is a monomorphism.
(2) If $u_{ji} = 0$ for all $1 \le i < j \le k$ then (2.5) and (2.6) hold trivially.
(3) If A_{k-1} is commutative and $a_{ki} = 1$ for all $1 \le i \le k-1$ then (2.5) and (2.6) hold.

Proof. (1) Note that β_i, ν_i are $\mathbb{F}$-linear for all $i = 1, \ldots, k$. Let $f = \sum_i a_i X_i \in A_{k-1}$, where $a_i \in \mathbb{F}$ and X_i are standard monomials for all i, and suppose that $\beta_k(f) = 0$. Then $\beta_k(X_i) = b_i X_i$ for some $0 \neq b_i \in \mathbb{F}$ by (2.9) and thus

$$0 = \beta_k(f) = \sum_i a_i b_i X_i.$$

It follows that all $a_i = 0$ since the standard monomials of A_k form an $\mathbb{F}$-basis. Thus $f = 0$.
(2) Trivial.
(3) Since A_{k-1} is commutative, $u_{ji} = 0$ and $a_{ji} = 1$ for all $1 \le i < j \le k-1$ and thus (2.5) and (2.6) hold. □

Theorem 2.6. *Let $A_k = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \ldots [x_k; \beta_k, \nu_k]$ be the iterated skew polynomial $\mathbb{F}$-algebra in Theorem 2.4. Suppose that $\mathbb{F}/(t-1)\mathbb{F}$ is isomorphic to $\mathbb{C}$, that $t-1$ is a nonunit and non-zero-divisor in A_k and that*

$$a_{ji} - 1 \in (t-1)\mathbb{F}, \quad \nu_j(x_i) \in (t-1)A_k \tag{2.14}$$

for all $1 \leq i < j \leq k$. *Then* $t-1$ *is a regular element of* A_k *and the semiclassical limit* $\overline{A}_k = A_k/(t-1)A_k$ *is Poisson isomorphic to an iterated Poisson polynomial* $\mathbb{C}$*-algebra*

$$\mathbb{C}[x_1][x_2;\alpha_2,\delta_2]_p \ldots [x_k;\alpha_k,\delta_k]_p,$$

where

$$\alpha_j(x_i) = \left(\frac{da_{ji}}{dt}|_{t=1}\right) x_i, \quad \delta_j(x_i) = \frac{d\nu_j(x_i)}{dt}|_{t=1} \tag{2.15}$$

for all $1 \leq i < j \leq k$. *(Derivatives are formal derivatives of power series in* $t-1$*.)*

Proof. Note that A_k is generated by $x_1, \ldots, x_k$ and that $t-1 \in \mathbb{F} \subset A_k$. Hence $t-1$ is a nonzero central element of A_k. Since

$$x_jx_i - x_ix_j = \beta_j(x_i)x_j + \nu_j(x_i) - x_ix_j = (a_{ji}-1)x_ix_j + \nu_j(x_i) \in (t-1)A_k \tag{2.16}$$

by (2.14), $\overline{A}_k$ is a commutative $\mathbb{C}$-algebra and thus $t-1$ is a regular element of A_k. Moreover we have

$$\begin{aligned}
\{\overline{x}_j, \overline{x}_i\} &= \overline{(t-1)^{-1}(x_jx_i - x_ix_j)} \\
&= \overline{\left(\frac{a_{ji}-1}{t-1}\right) x_ix_j + \left(\frac{\nu_j(x_i)}{t-1}\right)} && \text{(by (2.16))} \\
&= \left(\frac{da_{ji}}{dt}|_{t=1}\right) \overline{x}_i\overline{x}_j + \overline{\left(\frac{d\nu_j(x_i)}{dt}|_{t=1}\right)} && \text{(by (2.14))}
\end{aligned}$$

for all $1 \leq i < j \leq k$. Hence the result follows. □

For each positive integer k, we will write B_k for the commutative polynomial ring $\mathbb{C}[x_1, \ldots, x_k]$.

Lemma 2.7. *Let* $B_k = \mathbb{C}[x_1, \ldots, x_k]$ *be a Poisson algebra satisfying the following condition: for any* $1 \leq i < j \leq k$,

$$\{x_j, x_i\} = c_{ji}x_ix_j + p_{ji} \tag{2.17}$$

for some $c_{ji} \in \mathbb{C}, p_{ji} \in B_i$. *Then* B_k *is an iterated Poisson polynomial algebra of the form*

$$B_k = \mathbb{C}[x_1][x_2;\alpha_2,\delta_2]_p \ldots [x_k;\alpha_k,\delta_k]_p, \tag{2.18}$$

where

$$\alpha_j(x_i) = c_{ji}x_i, \qquad \delta_j(x_i) = p_{ji}.$$

Conversely, if B_k *is an iterated Poisson polynomial algebra of the form (2.18) then* B_k *is a Poisson algebra satisfying the condition (2.17).*

Proof. Suppose that B_k is a Poisson algebra satisfying the condition (2.17). Define derivations α_k, δ_k on B_{k-1} by

$$\alpha_k = \sum_{i=1}^{k-1} c_{ki}\frac{\partial}{\partial x_i}, \quad \delta_k = \sum_{i=1}^{k-1} p_{ki}\frac{\partial}{\partial x_i}.$$

Then α_k is a Poisson derivation, δ_k is a derivation and the pair (α_k, δ_k) satisfies (1.4) by [15, 1.1] since B_k is a Poisson algebra. Thus B_k is a Poisson polynomial algebra

$$B_k = \mathbb{C}[x_1, \ldots, x_{k-1}][x_k;\alpha_k,\delta_k]_p$$

over the Poisson subalgebra $B_{k-1} = \mathbb{C}[x_1, \ldots, x_{k-1}]$. The result follows by induction on k.

Conversely, if B_k is an iterated Poisson polynomial algebra of the form (2.18) then B_k is clearly a Poisson algebra satisfying the condition (2.17). □

Corollary 2.8. *Let B_k be an iterated Poisson polynomial $\mathbb{C}$-algebra*
$$B_k = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \dots [x_k; \alpha_k, \delta_k]_p$$
such that
$$\alpha_j(x_i) = c_{ji}x_i \ (c_{ji} \in \mathbb{C}), \quad \delta_j(x_i) \in \mathbb{C}[x_1, \dots, x_i]$$
for all $1 \le i < j \le k$ and let
$$a_{ji} \in \mathbb{F}, \quad u_{ji} \in \mathbb{F}[x_1, \dots, x_i]$$
such that
$$\begin{aligned} &a_{ji} - 1 \in (t-1)\mathbb{F}, && \frac{da_{ji}}{dt}|_{t=1} = c_{ji}, \\ &u_{ji} \in (t-1)\mathbb{F}[x_1, \dots, x_i], && \frac{du_{ji}}{dt}|_{t=1} = [\delta_j(x_i)], \end{aligned} \tag{2.19}$$
where $[\delta_j(x_i)]$ is the $\mathbb{C}$-linear combination of $\delta_j(x_i)$ by standard monomials of $x_1, \dots, x_i$. Set $A_1 = \mathbb{F}[x_1]$. Suppose that $\mathbb{F}/(t-1)\mathbb{F}$ is isomorphic to $\mathbb{C}$ and that $t-1$ is a nonunit and non-zero-divisor of an iterated skew polynomial $\mathbb{F}$-algebra
$$A_{k-1} = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \dots [x_{k-1}; \beta_{k-1}, \nu_{k-1}]$$
such that all β_j, ν_j satisfy (2.1) and (2.2). If $\mathbb{F}$-linear maps β_k, ν_k on A_{k-1} subject to (2.3) and (2.4) satisfy (2.5) and (2.6) then there exists an iterated skew polynomial $\mathbb{F}$-algebra
$$A_k = A_{k-1}[x_k; \beta_k, \nu_k] = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \dots [x_{k-1}; \beta_{k-1}, \nu_{k-1}][x_k; \beta_k, \nu_k]$$
and $t-1$ is a regular element of A_k such that B_k is Poisson isomorphic to the semiclassical limit $A_k/(t-1)A_k$.

Proof. By Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra $A_k = A_{k-1}[x_k; \beta_k, \nu_k]$. Since $t-1$ is still a nonunit and non-zero-divisor in A_k, it is a regular element of A_k and the semiclassical limit $A_k/(t-1)A_k$ is Poisson isomorphic to B_k by Theorem 2.6. □

3. Examples

In this section, we give examples which illustrate that A_k is an iterated skew polynomial $\mathbb{F}$-algebra such that $A_k/(t-1)A_k$ is Poisson isomorphic to a given Poisson algebra B_k. The first four examples appearing in [11] are Poisson Hopf algebras presented by iterated Poisson polynomial algebras. We are interested only in their Poisson structures because we have not found a formal way to give Hopf structures in their deformations yet.

Example 3.1. In [11, Example 3.2], $B = \mathbb{C}[x_1, x_2, x_3]$ is a Poisson algebra with the Poisson bracket
$$\{x_2, x_1\} = 0, \ \{x_3, x_1\} = \lambda_{11}x_1, \ \{x_3, x_2\} = \lambda_{21}x_1 + \lambda_{22}x_2,$$
where $\lambda_{\ell m} \in \mathbb{C}$. Observe that B is a Poisson polynomial $\mathbb{C}$-algebra
$$B = \mathbb{C}[x_1, x_2][x_3; \delta_3]_p,$$
where
$$\delta_3(x_1) = \lambda_{11}x_1, \ \delta_3(x_2) = \lambda_{21}x_1 + \lambda_{22}x_2.$$

Set $\mathbb{F} = \mathbb{C}[t]$ and

$$a_{31} = a_{32} = 1,\ u_{31} = f_{11}\lambda_{11}x_1 \in \mathbb{F}[x_1],\ u_{32} = f_{21}\lambda_{21}x_1 + f_{22}\lambda_{22}x_2 \in \mathbb{F}[x_1, x_2], \tag{3.1}$$

where $f_{\ell m} \in (t-1)\mathbb{F}$ with $\frac{df_{\ell m}}{dt}|_{t=1} = 1$, for example, $f_{\ell m} = (t-1)t^{N_{\ell m}}$ for some nonnegative integer $N_{\ell m}$. By Remark 2.5(3), the $\mathbb{F}$-linear maps β_3 and ν_3 on $\mathbb{F}[x_1, x_2]$ defined by

$$\beta_3(x_i) = x_i,\ \nu_3(x_i) = u_{3i}\ (i = 1, 2)$$

satisfy (2.5) and (2.6). Hence, by Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra

$$A = \mathbb{F}[x_1, x_2][x_3; \nu_3].$$

Moreover $t-1$ is a regular element of A and thus B is Poisson isomorphic to the semiclassical limit $A/(t-1)A$ of A by Corollary 2.8 since all a_{ji}, u_{ji} satisfy (2.19).

Example 3.2. In [11, Example 3.3], $B = \mathbb{C}[x_1, x_2, x_3, x_4]$ is a Poisson algebra with the Poisson bracket

$$\begin{aligned}
&\{x_2, x_1\} = \{x_3, x_1\} = \{x_3, x_2\} = 0,\\
&\{x_4, x_1\} = \lambda_{11}x_1,\\
&\{x_4, x_2\} = \lambda_{21}x_1 + \lambda_{22}x_2,\\
&\{x_4, x_3\} = \lambda_{31}x_1 + \lambda_{32}x_2 + (\lambda_{11} + \lambda_{22})x_3,
\end{aligned}$$

where $\lambda_{\ell m} \in \mathbb{C}$. Observe that B is a Poisson polynomial $\mathbb{C}$-algebra

$$B = \mathbb{C}[x_1, x_2, x_3][x_4; \delta_4]_p,$$

where

$$\delta_4(x_1) = \lambda_{11}x_1,\ \delta_4(x_2) = \lambda_{21}x_1 + \lambda_{22}x_2,\ \delta_4(x_3) = \lambda_{31}x_1 + \lambda_{32}x_2 + (\lambda_{11} + \lambda_{22})x_3.$$

Set $\mathbb{F} = \mathbb{C}[t]$ and

$$\begin{aligned}
&a_{41} = a_{42} = a_{43} = 1,\\
&u_{41} = f_{11}\lambda_{11}x_1 \in \mathbb{F}[x_1],\\
&u_{42} = f_{21}\lambda_{21}x_1 + f_{22}\lambda_{22}x_2 \in \mathbb{F}[x_1, x_2],\\
&u_{43} = f_{31}\lambda_{31}x_1 + f_{32}\lambda_{32}x_2 + (f_{11}\lambda_{11} + f_{22}\lambda_{22})x_3 \in \mathbb{F}[x_1, x_2, x_3],
\end{aligned} \tag{3.2}$$

where $f_{\ell m} \in (t-1)\mathbb{F}$ with $\frac{df_{\ell m}}{dt}|_{t=1} = 1$. By Remark 2.5(3), the $\mathbb{F}$-linear maps β_4 and ν_4 on $\mathbb{F}[x_1, x_2, x_3]$ subject to

$$\beta_4(x_i) = x_i,\ \nu_4(x_i) = u_{4i}\ (i = 1, 2, 3)$$

satisfy (2.5) and (2.6). Hence, by Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra

$$A = \mathbb{F}[x_1, x_2, x_3][x_4; \nu_4].$$

Moreover $t-1$ is a regular element of A and thus B is Poisson isomorphic to the semiclassical limit $A/(t-1)A$ of A by Corollary 2.8 since all a_{ji}, u_{ji} satisfy (2.19).

Example 3.3. In [11, Example 3.4], $C = \mathbb{C}[g^{\pm1}, x]$ is a Poisson algebra with the Poisson bracket

$$\{x, g\} = \lambda gx,$$

where $\lambda \in \mathbb{Z}$. Let $D = \mathbb{C}[g, h, x]$. Replacing g^{-1} in C by h in D, D is a Poisson algebra with the Poisson bracket

$$\{g, h\} = 0,\ \{x, g\} = \lambda gx,\ \{x, h\} = -\lambda hx,$$

namely $D = \mathbb{C}[g,h][x;\alpha]_p$ is a Poisson algebra by [15, 1.1], where $\alpha = \lambda g\frac{\partial}{\partial g} - \lambda h\frac{\partial}{\partial h}$ in $\mathbb{C}[g,h]$. Note that the ideal $(gh-1)D$ is a Poisson ideal such that $D/(gh-1)D$ is Poisson isomorphic to C.

Set $\mathbb{F} = \mathbb{C}[t,t^{-1}]$ and $a = t^{\lambda}$. By Remark 2.5(2) and Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra $A = \mathbb{F}[g,h][x;\beta]$ such that $gh-1$ is a central element in A, where

$$\beta(g) = ag,\ \beta(h) = a^{-1}h.$$

Set $B = A/(gh-1)A$ and note that $t-1$ is a regular element of A and B. The semiclassical limit $A/(t-1)A$ is Poisson isomorphic to D by Corollary 2.8 since

$$a-1 \in (t-1)\mathbb{F},\ \frac{da}{dt}|_{t=1} = \lambda,\ a^{-1}-1 \in (t-1)\mathbb{F},\ \frac{da^{-1}}{dt}|_{t=1} = -\lambda$$

and the semiclassical limit $B/(t-1)B$ is Poisson isomorphic to C.

Example 3.4. In [11, Example 3.7], $C = \mathbb{C}[E,F,K^{\pm 1}]$ is a Poisson algebra with the Poisson bracket

$$\begin{aligned}\{E,K\} &= -2KE,\\ \{F,K\} &= 2KF,\\ \{F,E\} &= \frac{1}{2}(K^{-1}-K).\end{aligned}$$

Set $D = \mathbb{C}[E,F,H,K]$. Replacing K^{-1} in C by H in D, it is observed that D is a Poisson algebra with Poisson bracket

$$\begin{array}{ll}\{H,K\} = 0, & \{E,H\} = 2HE,\\ \{E,K\} = -2KE, & \{F,H\} = -2HF,\\ \{F,K\} = 2KF, & \{F,E\} = \frac{1}{2}(H-K)\end{array}$$

and that the ideal $(HK-1)D$ is a Poisson ideal such that $D/(HK-1)D$ is Poisson isomorphic to C. In fact, D is an iterated Poisson polynomial $\mathbb{C}$-algebra

$$D = \mathbb{C}[H,K][E;\alpha_3]_p[F;\alpha_4,\delta_4]_p,$$

where

$$\begin{array}{lll}\alpha_3(H) = 2H, & \alpha_3(K) = -2K, & \\ \alpha_4(H) = -2H, & \alpha_4(K) = 2K, & \alpha_4(E) = 0,\\ \delta_4(H) = 0, & \delta_4(K) = 0, & \delta_4(E) = \frac{1}{2}(H-K).\end{array}$$

Set $\mathbb{F} = \mathbb{C}[t,t^{-1}]$ and $s = \sum_{i\geq 0}(1-t)^i \in \mathbb{C}[[t-1]]$. Since $ts = s-(1-t)s = 1$ in $\mathbb{C}[[t-1]]$, we have that $t^{-1} = s$ and thus $\mathbb{C}[t] \subset \mathbb{F} \subset \mathbb{C}[[t-1]]$. Set

$$\begin{array}{lllll}a_{31} = t^2, & a_{32} = t^{-2}, & a_{41} = t^{-2}, & a_{42} = t^2, & a_{43} = 1,\\ u_{31} = 0, & u_{32} = 0, & u_{41} = 0, & u_{42} = 0, & u_{43} = \frac{1}{4}(t-t^{-1})(H-K).\end{array} \tag{3.3}$$

Then there exists a skew polynomial $\mathbb{F}$-algebra $\mathbb{F}[H,K][E;\beta_3]$ by Remark 2.5(2) and, applying Theorem 2.4, there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A = \mathbb{F}[H,K][E;\beta_3][F;\beta_4,\nu_4],$$

where

$$\begin{aligned}
&\beta_3(H) = t^2H, && \beta_3(K) = t^{-2}K, && \\
&\beta_4(H) = t^{-2}H, && \beta_4(K) = t^2K, && \beta_4(E) = E, \\
&\nu_4(H) = 0, && \nu_4(K) = 0, && \nu_4(E) = \frac{1}{4}(t - t^{-1})(H - K).
\end{aligned}$$

Moreover the element $HK - 1$ is a central element of A and $t - 1$ is a regular element of A and $B = A/(HK - 1)A$. Note that the semiclassical limit $A/(t - 1)A$ is Poisson isomorphic to D by Corollary 2.8 since all a_{ji}, u_{ji} satisfy (2.19). Observe that the semiclassical limit $B/(t - 1)B$ is Poisson isomorphic to C.

Let $0, \pm 1 \neq q \in \mathbb{C}$. Then $t - q$ is a nonzero and nonunit in A and B. The deformation $B_q = B/(t - q)B$ is a nontrivial $\mathbb{C}$-algebra with the multiplication induced by that of B, which is isomorphic to $U_q(\mathfrak{sl}_2(\mathbb{C}))$ in [1, I.3.1] as shown in [8, 4.5].

Proposition 3.5. *Fix $h \in B_3 = \mathbb{C}[x_1, x_2, x_3]$ with degree ≤ 3. By* [9, Proposition 1.17], *B_3 becomes a Poisson algebra with Poisson bracket*

$$\{f, g\} = det \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{pmatrix} \tag{3.4}$$

for $f, g \in B_3$. Suppose that the Poisson bracket of B_3 satisfies the condition (2.17). Then h is of the form

$$h = \lambda x_1x_2x_3 + \mu x_3 + f_1x_2 + f_2,$$

where $\lambda, \mu \in \mathbb{C}$ and $f_1, f_2 \in \mathbb{C}[x_1]$ such that $\deg f_1 \leq 2$ *and* $\deg f_2 \leq 3$.

Proof. Note that the Poisson bracket of B_3 is as follows:

$$\{x_1, x_2\} = \frac{\partial h}{\partial x_3} \tag{3.5}$$

$$\{x_1, x_3\} = -\frac{\partial h}{\partial x_2} \tag{3.6}$$

$$\{x_2, x_3\} = \frac{\partial h}{\partial x_1}. \tag{3.7}$$

By (3.5) and (2.17), we have that $\frac{\partial h}{\partial x_3} = \{x_1, x_2\} = -c_{21}x_1x_2 - p_{21}$ and thus

$$h = -(c_{21}x_1x_2 + p_{21})x_3 + f, \tag{3.8}$$

where $c_{21} \in \mathbb{C}$, $p_{21} \in \mathbb{C}[x_1]$ with degree ≤ 2 and $f \in \mathbb{C}[x_1, x_2]$ with degree ≤ 3. By (3.6), (3.8) and (2.17), we have

$$c_{31}x_1x_3 + p_{31} = \{x_3, x_1\} = -c_{21}x_1x_3 + \frac{\partial f}{\partial x_2}$$

and thus $\frac{\partial f}{\partial x_2} = p_{31} \in \mathbb{C}[x_1]$. It follows that $f = p_{31}x_2 + f_2$ and thus

$$h = -(c_{21}x_1x_2 + p_{21})x_3 + p_{31}x_2 + f_2 \tag{3.9}$$

by (3.8), where $f_2 \in \mathbb{C}[x_1]$ such that $\deg p_{31} \leq 2$ and $\deg f_2 \leq 3$. By (3.7) and (3.9), we have that

$$-c_{32}x_2x_3 - p_{32} = \{x_2, x_3\} = -c_{21}x_2x_3 - p'_{21}x_3 + p'_{31}x_2 + f'_2,$$

where $p'_{21}=\frac{\partial p_{21}}{\partial x_1}, p'_{31}=\frac{\partial p_{31}}{\partial x_1}, f'_2=\frac{\partial f_2}{\partial x_1}$, and thus $p_{21}\in\mathbb{C}$. Hence h is of the form

$$h=\lambda x_1x_2x_3+\mu x_3+f_1x_2+f_2$$

for some $\lambda,\mu\in\mathbb{C}$ and $f_1,f_2\in\mathbb{C}[x_1]$ with $\deg f_1\le 2$ and $\deg f_2\le 3$, as claimed. $\square$

Example 3.6. Retain the notations of Proposition 3.5. Suppose that $\deg f_1=0$, namely $f_1\in\mathbb{C}$. By (3.4), B_3 is a Poisson algebra with the Poisson bracket

$$\{x_2,x_1\}=-\lambda x_1x_2-\mu,\ \{x_3,x_1\}=\lambda x_1x_3+f_1,\ \{x_3,x_2\}=-\lambda x_2x_3-\frac{\partial f_2}{\partial x_1}.$$

Hence B_3 is an iterated Poisson polynomial $\mathbb{C}$-algebra

$$B_3=\mathbb{C}[x_1][x_2;\alpha_2,\delta_2]_p[x_3;\alpha_3,\delta_3]_p$$

by [15, 1.1], where

$$\begin{aligned}&\alpha_2(x_1)=-\lambda x_1, && \alpha_3(x_1)=\lambda x_1, && \alpha_3(x_2)=-\lambda x_2,\\ &\delta_2(x_1)=-\mu, && \delta_3(x_1)=f_1, && \delta_3(x_2)=-\frac{\partial f_2}{\partial x_1}.\end{aligned}$$

Let $\mathbb{F}=\mathbb{C}[[t-1]]$ and let $U(\mathbb{F})$ be the unit group of $\mathbb{F}$. Note that $t-1$ is a nonzero, nonunit and non-zero-divisor of $\mathbb{F}$. Fix $\widetilde{\lambda}\in U(\mathbb{F})$, $\widetilde{\mu},\widetilde{f_1}\in\mathbb{F}$, $\widetilde{g}\in\mathbb{F}[x_1]$ such that

$$\begin{aligned}&\widetilde{\lambda}-1\in(t-1)\mathbb{F}, && \widetilde{\mu},\ \widetilde{f_1}\in(t-1)\mathbb{F}, && \widetilde{g}\in(t-1)\mathbb{F}[x_1],\\ &\frac{d\widetilde{\lambda}}{dt}|_{t=1}=\lambda, && \frac{d\widetilde{\mu}}{dt}|_{t=1}=\mu,\ \frac{d\widetilde{f_1}}{dt}|_{t=1}=f_1, && \frac{d\widetilde{g}}{dt}|_{t=1}=\frac{\partial f_2}{\partial x_1}.\end{aligned}\tag{3.10}$$

(Such ones exist. For example, $\widetilde{\lambda}=e^{\lambda(t-1)}$, $\widetilde{\mu}=(t-1)\mu$, $\widetilde{f_1}=(t-1)f_1$, $\widetilde{g}=(t-1)\frac{\partial f_2}{\partial x_1}$.) Set

$$a_{21}=\widetilde{\lambda}^{-1},\ u_{21}=-\widetilde{\mu}.\tag{3.11}$$

The $\mathbb{F}$-linear maps β_2 and ν_2 on $\mathbb{F}[x_1]$ defined by

$$\beta_2(x_1)=a_{21}x_1=\widetilde{\lambda}^{-1}x_1,\ \ \nu_2(x_1)=u_{21}=-\widetilde{\mu}$$

satisfy (2.5) and (2.6) trivially. Hence there exists a skew polynomial $\mathbb{F}$-algebra $A_2=\mathbb{F}[x_1][x_2;\beta_2,\nu_2]$ by Theorem 2.4.

Set

$$\begin{aligned}a_{31}&=\widetilde{\lambda}, & a_{32}&=\widetilde{\lambda}^{-1},\\ u_{31}&=\widetilde{f_1}, & u_{32}&=-\widetilde{g}.\end{aligned}\tag{3.12}$$

Since $u_{21},u_{31}\in\mathbb{F}$, $u_{32}\in\mathbb{F}[x_1]$ and $a_{31}^{-1}=a_{21}=a_{32}$, the $\mathbb{F}$-linear maps β_3 and ν_3 on A_2 subject to

$$\begin{aligned}&\beta_3(x_1)=a_{31}x_1=\widetilde{\lambda}x_1, && \beta_3(x_2)=a_{32}x_2=\widetilde{\lambda}^{-1}x_2,\\ &\nu_3(x_1)=u_{31}=\widetilde{f_1}, && \nu_3(x_2)=u_{32}=-\widetilde{g}\end{aligned}$$

satisfy (2.5) and (2.6). Hence, by Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra

$$A_3=A_2[x_3;\beta_3,\nu_3]=\mathbb{F}[x_1][x_2;\beta_2,\nu_2][x_3;\beta_3,\nu_3].$$

Note that $t-1$ is a regular element in A_3. Thus the semiclassical limit $A_3/(t-1)A_3$ is Poisson isomorphic to B_3 by Corollary 2.8 since all a_{ji}, u_{ji} satisfy (2.19) by (3.10).

For every $1 \neq q \in \mathbb{C}$, $t-q$ is a unit in $\mathbb{F} = \mathbb{C}[[t-1]]$ and thus $A_3/(t-q)A_3$ is trivial. Hence, in order to find nontrivial deformations, we need a suitable subalgebra A_3' of A_3 such that deformations $A_3'/(t-q)A_3'$ are nontrivial, as one sees below.

As a special case, let $\mathbb{F} = \mathbb{C}[t, t^{-1}]$ and $\lambda = -2,\ \mu = 2,\ f_1 = 2,\ f_2 = 2x_1$. Then

$$h = -2x_1x_2x_3 + 2x_3 + 2x_2 + 2x_1$$

and B_3 is a Poisson $\mathbb{C}$-algebra with the Poisson bracket

$$\{x_2, x_1\} = 2x_1x_2 - 2,\ \{x_3, x_1\} = -2x_1x_3 + 2,\ \{x_3, x_2\} = 2x_2x_3 - 2.$$

Setting

$$\widetilde{\lambda} = t^{-2},\ \widetilde{\mu} = t^2 - 1,\ \widetilde{f_1} = -(t^{-2} - 1),\ \widetilde{g} = t^2 - 1,$$

there is an $\mathbb{F}$-algebra $A_3 = \mathbb{F}[x_1][x_2; \beta_2, \nu_2][x_3; \beta_3, \nu_3]$ such that

$$\begin{aligned} &\beta_2(x_1) = a_{21}x_1 = t^2x_1, && \beta_3(x_1) = a_{31}x_1 = t^{-2}x_1, && \beta_3(x_2) = a_{32}x_2 = t^2x_2, \\ &\nu_2(x_1) = u_{21} = -(t^2 - 1), && \nu_3(x_1) = u_{31} = -(t^{-2} - 1), && \nu_3(x_2) = u_{32} = -(t^2 - 1). \end{aligned}$$

Note that A_3 is the $\mathbb{F}$-algebra generated by x_1, x_2, x_3 subject to the relations

$$t^2x_1x_2 - x_2x_1 = t^2 - 1,\ t^2x_3x_1 - x_1x_3 = t^2 - 1,\ t^2x_2x_3 - x_3x_2 = t^2 - 1. \tag{3.13}$$

Let $0, 1 \neq q \in \mathbb{C}$ and let A_3^q be the deformation $A_3^q = A_3/(t-q)A_3$ of B_3. Then A_3^q is the $\mathbb{C}$-algebra generated by x_1, x_2, x_3 subject to the relations obtained from (3.13) by replacing t by q. Observe that the set $\{x_3^i | i = 0, 1, \ldots\}$ is an Ore set of A_3^q by the second and the third equations of (3.13). The localization $A_3^q[x_3^{-1}]$ of A_3^q at $\{x_3^i | i = 0, 1, \ldots\}$ is isomorphic to $U_q(\mathfrak{sl}_2)$ by Ito, Terwilliger and Weng [16], which is Y_q in [8, 4.5].

Example 3.7. As in [5, 2.2], we find a quantization and deformations of a well-known Poisson algebra $B_k = \mathbb{C}[x_1, x_2, \ldots, x_{2k-1}, x_{2k}]$ with Poisson bracket

$$\{f, g\} = \sum_{i=1}^{k} \left(-\frac{\partial f}{\partial x_{2i-1}} \frac{\partial g}{\partial x_{2i}} + \frac{\partial g}{\partial x_{2i-1}} \frac{\partial f}{\partial x_{2i}} \right),$$

which is called Poisson Weyl algebra in [2, 1.1.A] and [14, 1.3]. Since B_k is a Poisson algebra with Poisson bracket

$$\{x_j, x_i\} = \begin{cases} 1, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases}$$

for $j > i$, B_k is an iterated Poisson polynomial algebra

$$B_k = \mathbb{C}[x_1][x_2; \delta_2]_p \ldots [x_{2k-1}]_p[x_{2k}; \delta_{2k}]_p,$$

where

$$\delta_{2\ell}(x_i) = \begin{cases} 1, & \text{if } i = 2\ell - 1, \\ 0, & \text{if } i \neq 2\ell - 1. \end{cases}$$

Set $\mathbb{F} = \mathbb{C}[t]$ and let

$$a_{ji} = 1,\ u_{ji} = \begin{cases} t - 1, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.14}$$

for all $1 \leq i < j \leq 2k$. By Theorem 2.4, there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A_k = \mathbb{F}[x_1][x_2; \nu_2] \ldots [x_{2k-1}][x_{2k}; \nu_{2k}],$$

where

$$\nu_{2\ell}(x_i) = \begin{cases} t-1, & \text{if } i = 2\ell - 1, \\ 0, & \text{if } i \neq 2\ell - 1. \end{cases}$$

Thus A_k is an $\mathbb{F}$-algebra generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$x_j x_i - x_i x_j = \begin{cases} t-1 & \text{if } j = 2\ell, i = 2\ell - 1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.15}$$

which is the algebra appearing in [13, Proposition 3.2]. For each $0 \neq \lambda \in \mathbb{C}$, a deformation $A_\lambda = A_k/(t-1-\lambda)A_k$ is a $\mathbb{C}$-algebra generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$x_j x_i - x_i x_j = \begin{cases} \lambda, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we get a family of infinite nontrivial deformations $\{A_\lambda | 0 \neq \lambda \in \mathbb{C}\}$, all of which are isomorphic to the k-th Weyl algebra by [13, Proposition 3.4].

Note that $t-1$ is a regular element of A_k. By Corollary 2.8, the semiclassical limit $A_k/(t-1)A_k$ is Poisson isomorphic to B_k since

$$a_{ji} - 1 \in (t-1)\mathbb{F},\ \frac{da_{ji}}{dt}|_{t=1} = 0,\ u_{ji} \in (t-1)A_i,\ \frac{du_{ji}}{dt}|_{t=1} = [\delta_j(x_i)].$$

Example 3.8. Let B_k be the Poisson Weyl algebra given in Example 3.7. Set $\mathbb{F} = \mathbb{C}[[t-1]]$ and

$$a_{ji} = \begin{cases} \cos(t-1), & \text{if } i+j \text{ is odd,} \\ \sec(t-1), & \text{if } i+j \text{ is even,} \end{cases} \qquad u_{ji} = \begin{cases} \sin(t-1), & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.16}$$

for all $1 \leq i < j \leq 2k$. Note that $a_{ji}, u_{ji} \in \mathbb{F}$ by elementary calculus.

We will show by induction on k that there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A_k = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \ldots [x_{2k-1}; \beta_{2k-1}][x_{2k}; \beta_{2k}, \nu_{2k}],$$

where

$$\beta_j(x_i) = a_{ji} x_i,\ \nu_j(x_i) = u_{ji}$$

for all $1 \leq i < j \leq 2k$. If $k = 1$ then there exists the skew polynomial $\mathbb{F}$-algebra $A_1 = \mathbb{F}[x_1][x_2; \beta_2, \nu_2]$ trivially by Theorem 2.4. Suppose that $k > 1$ and assume that there exists an iterated skew polynomial $\mathbb{F}$-algebra A_{k-1}. Note that, for any positive integers i, j, ℓ,

$$\begin{aligned} &i + j \text{ is odd if and only if} \\ &\qquad (\ell + j \text{ is odd and } \ell + i \text{ is even}) \text{ or } (\ell + j \text{ is even and } \ell + i \text{ is odd}). \end{aligned} \tag{3.17}$$

Observe that $\mathbb{F}$-linear maps β_{2k-1} and ν_{2k-1} satisfy (2.6) trivially since $\nu_{2k-1}(u_{ji}) = 0$ and $u_{2k-1,i} = 0$ for all $1 \leq i < 2k-1$ and that they also satisfy (2.5) by (3.17) since $\beta_{2k-1}(u_{ji}) = u_{ji}$. Hence there exists a skew polynomial $\mathbb{F}$-algebra $A_{k-1}[x_{2k-1}; \beta_{2k-1}]$ by Theorem 2.4. For $\mathbb{F}$-linear maps β_{2k} and ν_{2k}, they satisfy (2.5) and (2.6) by (3.17) since $\beta_{2k}(u_{ji}) = u_{ji}$ and $\nu_{2k}(u_{ji}) = 0$ and thus there exists $A_k = A_{k-1}[x_{2k-1}; \beta_{2k-1}][x_{2k}; \beta_{2k}, \nu_{2k}]$ by Theorem 2.4.

Note that $t-1$ is a regular element of A_k. By Corollary 2.8, the semiclassical limit $A_k/(t-1)A_k$ is Poisson isomorphic to B_k since

$$a_{ji} - 1 \in (t-1)\mathbb{F},\ \frac{da_{ji}}{dt}|_{t=1} = 0,\ u_{ji} \in (t-1)A_i,\ \frac{du_{ji}}{dt}|_{t=1} = [\delta_j(x_i)]$$

by elementary calculus.

Note that A_k is an $\mathbb{F}$-algebra generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$
\begin{aligned}
&x_{2\ell}x_{2\ell-1} - \cos(t-1)x_{2\ell-1}x_{2\ell} = \sin(t-1), && (\ell = 1, \ldots, k),\\
&x_jx_i - \sec(t-1)x_ix_j = 0, && (i < j,\ \ i+j \text{ is even}),\\
&x_jx_i - \cos(t-1)x_ix_j = 0, && \begin{pmatrix} i < j,\ i+j \text{ is odd},\\ \text{if } j = 2\ell \text{ then } i \neq 2\ell - 1\end{pmatrix}.
\end{aligned}
\tag{3.18}
$$

For any $0 \neq q \in \mathbb{C}$, $t - 1 - q$ is a unit in A_k and thus $A_k/(t-1-q)A_k$ is trivial. It follows that we need an appropriate subalgebra of A_k to find a nontrivial deformation. For instance, let A'_k be the $\mathbb{C}[t]$-subalgebra of A_k generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$. Evaluating A'_k to π at $t-1$, we have a deformation A^π_k which is the $\mathbb{C}$-algebra generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$x_jx_i + x_ix_j = 0 \ (j > i)$$

by (3.18). In this case the evaluation map φ from A'_k onto A^π_k defined by $f \mapsto f|_{t-1=\pi}$ is a $\mathbb{C}$-algebra epimorphism and thus $A'_k/\ker\varphi \cong A^\pi_k$.

Example 3.9. The commutative $\mathbb{C}$-algebra $B = \mathbb{C}[x_1, \ldots, x_n]$ is a Poisson $\mathbb{C}$-algebra with Poisson bracket

$$\{x_j, x_i\} = x_ix_j$$

for all $1 \leq i < j \leq n$ by [4, Example 4.5]. Note that B is an iterated Poisson polynomial $\mathbb{C}$-algebra

$$B = \mathbb{C}[x_1][x_2; \alpha_2]_p \ldots [x_n; \alpha_n]_p,$$

where $\alpha_j(x_i) = x_i$ for all $1 \leq i < j \leq n$.

Set $\mathbb{F} = \mathbb{C}[t]$ and $a_{ki} = t$ for $1 \leq i < k \leq n$. Then, by Remark 2.5(2) and Theorem 2.4, there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A = \mathbb{F}[x_1][x_2; \beta_2] \ldots [x_n; \beta_n],$$

where $\beta_k(x_i) = a_{ki}x_i$ for all $1 \leq i < k \leq n$. Note that $t - 1$ is a regular element of A. By Corollary 2.8, $A/(t-1)A$ is Poisson isomorphic to B since

$$a_{ki} - 1 \in (t-1)\mathbb{F},\ \frac{da_{ki}}{dt}|_{t=1} = 1.$$

Let $0, 1 \neq q \in \mathbb{C}$. The deformation $A_q = A/(t-q)A$ of B is the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

$$x_jx_i = qx_ix_j$$

for all $1 \leq i < j \leq n$, which is the coordinate ring $\mathcal{O}_q(\mathbb{C}^n)$ of quantum affine n-space in [1, I.2.1].

Example 3.10. A Poisson 2×2-matrices algebra is the coordinate ring of 2×2-matrices, $\mathcal{O}(M_2(\mathbb{C})) = \mathbb{C}[x, y, z, w]$, with Poisson bracket

$$
\begin{aligned}
&\{x, y\} = xy, && \{x, z\} = xz, && \{x, w\} = 2yz,\\
&\{y, z\} = 0, && \{y, w\} = yw, && \{z, w\} = zw
\end{aligned}
$$

by [4, Example 4.9]. Note that $\mathcal{O}(M_2(\mathbb{C}))$ is an iterated Poisson polynomial $\mathbb{C}$-algebra

$$\mathcal{O}(M_2(\mathbb{C})) = \mathbb{C}[y][z][x; \alpha_3]_p[w; \alpha_4, \delta_4]_p,$$

where

$$\begin{aligned}
&\alpha_3(y)=y, &&\alpha_3(z)=z,\\
&\alpha_4(y)=-y, &&\alpha_4(z)=-z, &&\alpha_4(x)=0,\\
&\delta_4(y)=0, &&\delta_4(z)=0, &&\delta_4(x)=-2yz.
\end{aligned}$$

Set $\mathbb{F}=\mathbb{C}[t,t^{-1}]$ and

$$\begin{aligned}
&a_{31}=t, &&a_{32}=t,\\
&u_{31}=0, &&u_{32}=0,\\
&a_{41}=a_{31}^{-1}, &&a_{42}=a_{32}^{-1}, &&a_{43}=1,\\
&u_{41}=0, &&u_{42}=0, &&u_{43}=-(t-t^{-1})yz.
\end{aligned}\tag{3.19}$$

We show that there exists an iterated skew polynomial $\mathbb{F}$-algebra

$$A=\mathbb{F}[y,z][x;\beta_3][w;\beta_4,\nu_4],$$

where

$$\begin{aligned}
&\beta_3(y)=a_{31}y, &&\beta_3(z)=a_{32}z,\\
&\beta_4(y)=a_{31}^{-1}y, &&\beta_4(z)=a_{32}^{-1}z, &&\beta_4(x)=a_{43}x,\\
&\nu_4(y)=0, &&\nu_4(z)=0, &&\nu_4(x)=u_{43}.
\end{aligned}\tag{3.20}$$

By Remark 2.5(2) and Theorem 2.4, there exists a skew polynomial $\mathbb{F}$-algebra $\mathbb{F}[y,z][x;\beta_3]$. Note that $\mathbb{F}[y,z]$ is commutative and $u_{43}\in\mathbb{F}[y,z]$, $a_{42}a_{32}=a_{41}a_{31}=1$. Hence $\mathbb{F}$-linear maps β_4 and ν_4 satisfy (2.5) and (2.6) and thus there exists an iterated skew polynomial $\mathbb{F}$-algebra A by Theorem 2.4. Note that $t-1$ is a regular element of A. Hence the semiclassical limit $A/(t-1)A$ is Poisson isomorphic to $\mathcal{O}(M_2(\mathbb{C}))$ by Corollary 2.8 since all a_{ji}, u_{ji} satisfy (2.19).

The deformation

$$A_q=A/(t-q)A,\ \ (0,1\neq q\in\mathbb{C})$$

with multiplication induced by that of A is the $\mathbb{C}$-algebra generated by x,y,z,w subject to the relations

$$zy=yz,\qquad xy=qyx,\qquad xz=qzx,$$
$$yw=qwy,\quad zw=qwz,\quad xw-wx=(q-q^{-1})yz.$$

Following [1, I.1.7], A_q is the quantum 2×2-matrices algebra $\mathcal{O}_q(M_2(\mathbb{C}))$ as expected.

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