Poisson Algebras I, Non-commutative Algebras

Maram Alossaimi (Supervisor: Prof. Vladimir Bavula)

School of Mathematics and Statistics, malossaimi1@sheffield.ac.uk\maram.alosaimi@gmail.com

1. Introduction

A (commutative) algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\}: D\times D\to D$, called a *Poisson bracket*, such that
1. $\{a,b\}=-\{b,a\}$ for all $a,b\in D$ (anti-commutative),
2. $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0$ for all $a,b,c\in D$ (Jacobi identity), and
3. $\{ab,c\}=a\{b,c\}+\{a,c\}b$ for all $a,b,c\in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D,I\}\subseteq I$. We denote by $\langle a\rangle$ the Poisson ideal of D generated by the element a. Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ\subseteq P\Rightarrow I\subseteq P \quad\text{or}\quad J\subseteq P,$$

where I and J are Poisson ideals of D. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K. A K-linear map $\alpha: D\to D$ is a *Poisson derivation* of D if α is a K-derivation of D and

$$\alpha(\{a,b\})=\{\alpha(a),b\}+\{a,\alpha(b)\}\quad\text{for all } a,b\in D.$$

A set of all Poisson derivations of D is denoted by $\mathrm{PDer}_K(D)$.

2. How did we get our class of Poisson algebras $\mathcal{A}$?

Lemma. [Oh] *Let D be a Poisson algebra over a field K, $c\in K$, $u\in D$ and $\alpha,\beta\in\mathrm{PDer}_K(D)$ such that*

$$\alpha\beta=\beta\alpha \quad\textit{and}\quad \{d,u\}=(\alpha+\beta)(d)u \quad\textit{for all } d\in D. \tag{1}$$

Then the polynomial ring $D[x,y]$ becomes a Poisson algebra with Poisson bracket

$$\{d,y\}=\alpha(d)y,\quad \{d,x\}=\beta(d)x \quad\textit{and}\quad \{y,x\}=cyx+u \quad\textit{for all } d\in D. \tag{2}$$

The Poisson algebra $D[x,y]$ with Poisson bracket (2) is denoted by $(D;\alpha,\beta,c,u)$.

3. How did we construct $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A}=(K[t];\alpha,\beta,c,u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a,b\}=0$ for all $a,b\in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0=\{d,u\}=(\alpha+\beta)(d)u \quad\text{for all } d\in K[t],$$

which implies that precisely one of the three classes holds:

(Class I: $\alpha+\beta=0$ and $u=0$), (Class II: $\alpha+\beta=0$ and $u\neq0$) or (Class III: $\alpha+\beta\neq0$ and $u=0$).

4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha,\beta\in\mathrm{PDer}_K=\mathrm{Der}_K(K[t])=K[t]\partial_t$ such that $\alpha=f\partial_t$ and $\beta=g\partial_t$, where $f,\ g\in K[t]\backslash\{0\}$, $\partial_t=d/dt$ then*

$$\alpha\beta=\beta\alpha \quad\textit{if and only if}\quad g=\frac{1}{\lambda}f \quad\textit{for some}\quad \lambda\in K^\times:=K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha=f\partial_t$, $\beta=\lambda^{-1}f\partial_t$, $c\in K$, $u\in K[t]$, where $f\in K[t]$ and $\lambda\in K^\times$. Then we have the class of Poisson algebras $\mathcal{A}=K[t][x,y]=(K[t];\alpha=f\partial_t,\beta=\lambda^{-1}f\partial_t,c,u)$ with Poisson bracket defined by the rule:

$$\{t,y\}=fy,\qquad \{t,x\}=\lambda^{-1}fx \quad\text{and}\quad \{y,x\}=cyx+u. \tag{4}$$

The next diagram shows the first class (Class I) of Poisson algebras $\mathcal{A}$.

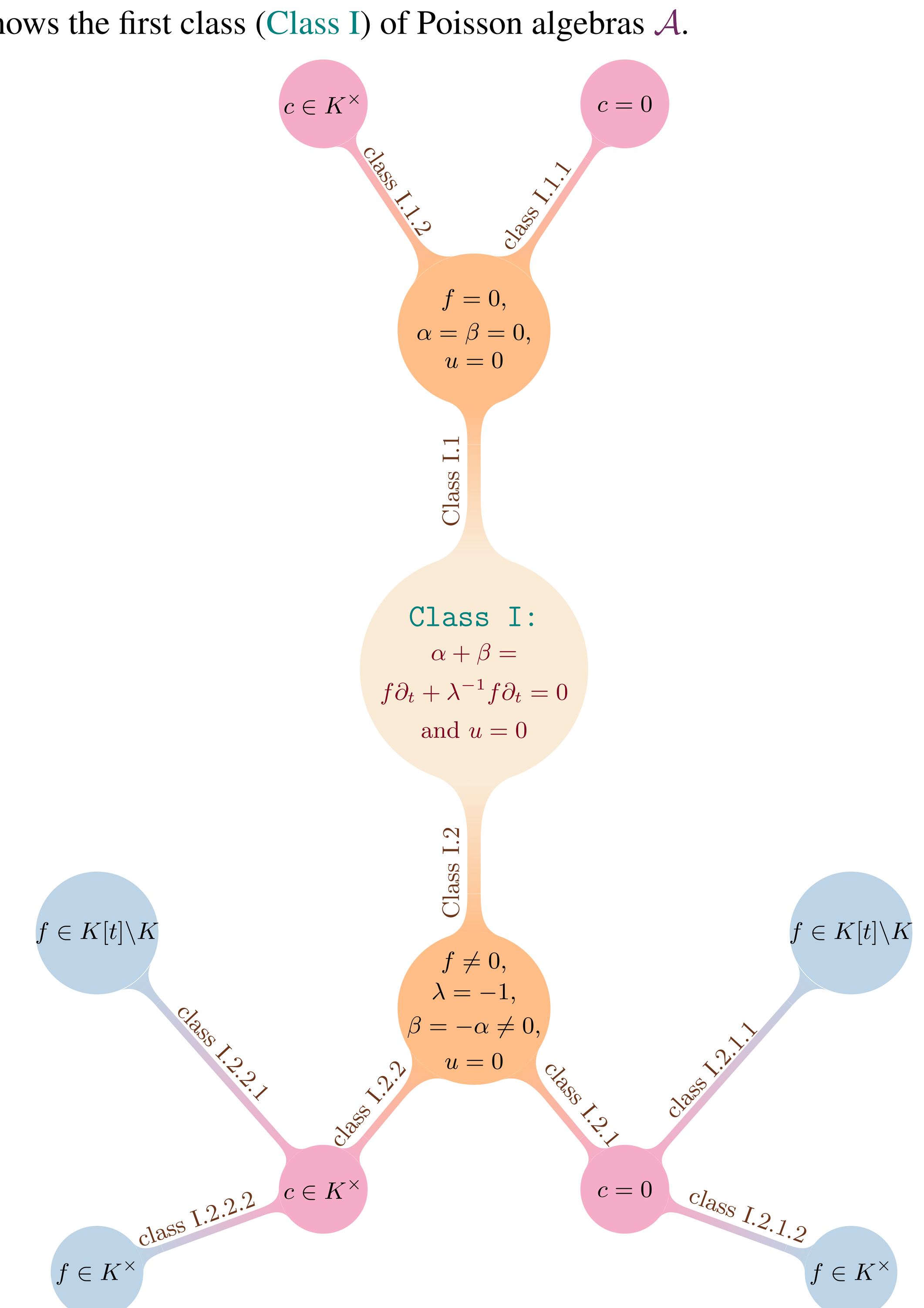

Diagram 1: `Structure of the first class of Poisson algebras` $\mathcal{A}$

Class I: $\alpha+\beta=f\partial_t+\frac{1}{\lambda}f\partial_t=(1+\frac{1}{\lambda})f\partial_t=0$ and $u=0$

Class I.1:
If $f=0$, i.e. $\alpha=\beta=0$ and $u=0$ then $\mathcal{A}_1=(K[t];0,0,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0\quad\text{and}\quad \{y,x\}=cyx. \tag{5}$$

Class `I.1.1`: If $c=0$ then the polynomial Poisson algebra $\mathcal{A}_2=(K[t];0,0,0,0)$ has trivial Poisson structure and $\mathrm{PSpec}(\mathcal{A}_2)$ is the spectrum of the polynomial ring in three variables, i.e. $\mathrm{Spec}(K[t,x,y])$.

Class `I.1.2`: If $c\in K^\times$ then $\mathcal{A}_3=(K[t];0,0,c,0)$ is a Poisson algebra with Poisson bracket (5). The Poisson spectrum of $\mathcal{A}_3$ is in the below diagram, 2.

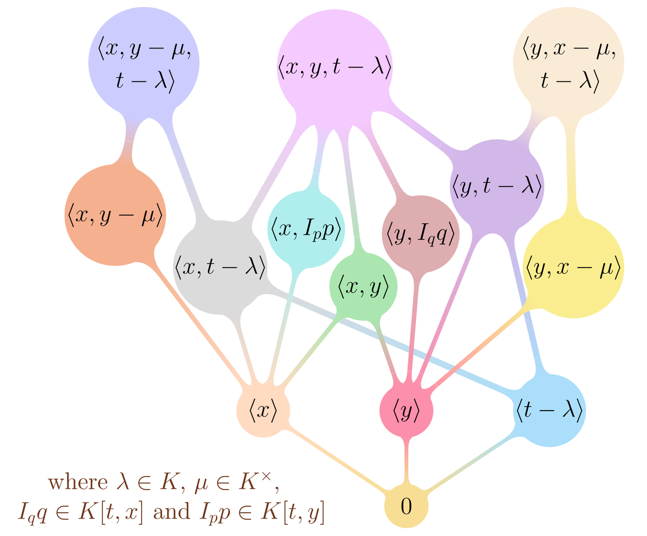

Diagram 2: `The containment information between Poisson prime ideals of` $\mathcal{A}_3$

Class I.2:
If $\lambda=-1$, i.e. $\beta=-\alpha=-f\partial_t$ for some $f\in K[t]\backslash\{0\}$ and $u=0$ then $\mathcal{A}_4=(K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=fy,\qquad \{t,x\}=-fx\quad\text{and}\quad \{y,x\}=cyx. \tag{6}$$

Class `I.2.1`: If $c=0$ then $\mathcal{A}_5=(K[t];f\partial_t,-f\partial_t,0,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=fy,\qquad \{t,x\}=-fx\quad\text{and}\quad \{y,x\}=0. \tag{7}$$

Class `I.2.1.1`:
If $f\in K[t]\backslash K$ and $R_f=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_6=(K[t];f\partial_t,-f\partial_t,0,0)$ is a Poisson algebra with Poisson bracket (7). The Poisson spectrum of $\mathcal{A}_6$ is in the below diagram, 3.

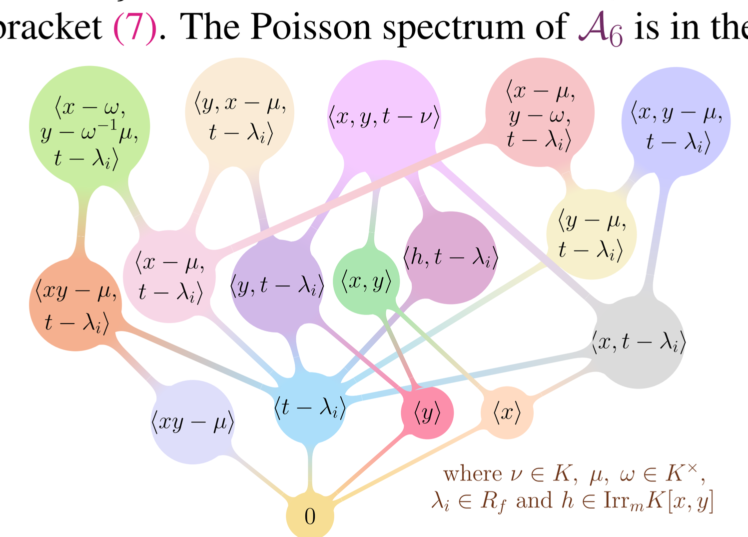

Diagram 3: `The containment information between Poisson prime ideals of` $\mathcal{A}_6$

Class `I.2.1.2`:
If $f=a\in K^\times$, i.e. $R_a=\emptyset$ then $\mathcal{A}_7=(K[t];a\partial_t,-a\partial_t,0,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=ay,\qquad \{t,x\}=-ax\quad\text{and}\quad \{y,x\}=0. \tag{8}$$

The Poisson spectrum of $\mathcal{A}_7$ is in the below diagram, 4.

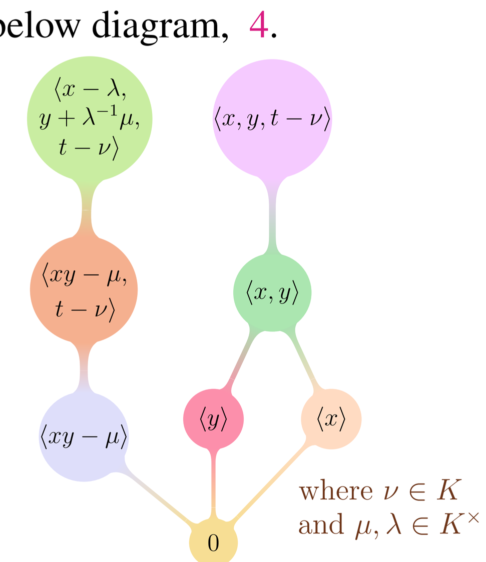

Diagram 4: `The containment information between Poisson prime ideals of` $\mathcal{A}_7$

Class `I.2.2`: If $c\in K^\times$ then $\mathcal{A}_8=(K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket (6).

Class `I.2.2.1`:
If $f\in K[t]\backslash K$ and $R_f=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_9=(K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket (6). The Poisson spectrum of $\mathcal{A}_9$ is in the below diagram, 5.

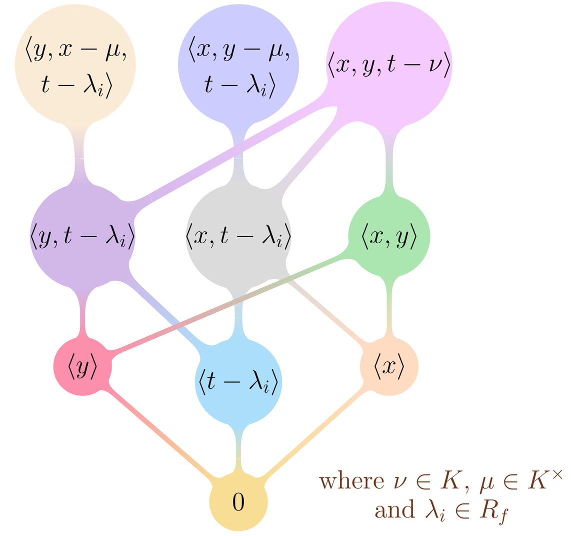

Diagram 5: `The containment information between Poisson prime ideals of` $\mathcal{A}_9$

Class `I.2.2.2`:
If $f=a\in K^\times$, i.e. $R_a=\emptyset$ then $\mathcal{A}_{10}=(K[t];a\partial_t,-a\partial_t,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=ay,\qquad \{t,x\}=-ax\quad\text{and}\quad \{y,x\}=cyx. \tag{9}$$

The Poisson spectrum of $\mathcal{A}_{10}$ is a subset of $\mathrm{PSpec}(\mathcal{A}_9)$.

5. Conclusion / Future research

A classification of Poisson prime ideals of Poisson algebras $\mathcal{A}$ was obtained in 10 classes out of 26. We will complete the classification. Then we aim to classify some simple finite dimensional Poisson modules over $\mathcal{A}$.

Acknowledgements

I would like to thank my supervisor Vladimir for providing guidance and feedback throughout this research. Also, I would like to thank my sponsor the University of Imam Mohammad Ibn Saud Islamic.

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