Lecture 1: Poisson-Lie groups and Lie bialgebras

Def. A Poisson-Lie group is a Lie group G with a Poisson bracket $\{\,,\}$ such that the multiplication map $G\times G \longrightarrow G$ is Poisson.

(same definition is made for complex Lie gps, algebraic groups, formal groups).

This means: if f, g regular functions on G then

$$\{f,g\}(xy) = \{f,g\}_x(xy) + \{f,g\}_y(xy).$$

In terms of Poisson bivector Π of G ($\Pi \in \Gamma(G, \Lambda^2 TG)$)

$$(1) \quad \Pi(xy) = \Pi(x)y + x\Pi(y)$$

In particular, setting $x=y=e$

we get

$$\Pi(e) = \Pi(e) + \Pi(e) \Rightarrow \Pi(e) = 0.$$

Thus e is a symplectic leaf of G.

Poisson-Lie groups form a category $PLG_{\mathbb{R}}$: objects = (real) Poisson-Lie groups, morphisms = Poisson homomorphisms.

Similarly can define $PLG_{\mathbb{C}}$.

__Prop.__ The inversion map $i : G \to G$ of a Poisson-Lie group G is anti-Poisson, i.e.

$$\{i^*f, i^*g\} = -i^*\{f, g\}.$$

__Pf.__ Set $y = x^{-1}$ in (1). We get

$$\Pi(x)x^{-1} + x\Pi(x^{-1}) = \Pi(e) = 0$$

$$\Rightarrow \Pi(x^{-1}) = -x^{-1}\Pi(x)x^{-1} \qquad (2)$$

But $d_x i = -\lambda_{x^{-1}*} \circ \rho_{x^{-1}*} : T_x G \to T_{x^{-1}} G$
(composition of left and right translation by x^{-1} with a minus sign)
(as $\frac{d}{dt} x(t)^{-1} = -x(t)^{-1} \frac{dx(t)}{dt} \cdot x(t)^{-1}$).

Thus (2) means exactly that i is anti-Poisson $\boxtimes$

The main tool of studying Lie groups is passing to the Lie algebra, and in fact the fundamental theorems of Lie theory tell us that the category of simply connected Lie groups is equivalent to the category of finite dimensional Lie algebras.

So let us see what structure on Lie G is induced by a Poisson-Lie structure on G. This leads us to the notion of a Lie bialgebra.

Let X be a Poisson manifold with $e \in X$ such that $\Pi(e)=0$. We claim that in this case $\mathfrak{g} = T_e^*X$ has a natural Lie algebra structure. Indeed, in this case the maximal ideal $I \in C^\infty(X)$ of functions vanishing at e is closed under the Poisson bracket, so it is a Lie algebra, and $I^2 \subset I$ is an ideal in this Lie algebra, so $T_e^*X = I/I^2$ is a Lie algebra. I.e. the linear approximation of a

Poisson manifold X near a zero e of the Poisson bracket is the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g} = T_e^* X$

In particular, this shows that if G is a Poisson Lie group then not only $\mathfrak{g} = \mathrm{Lie}\, G$ a Lie algebra, but also $\mathfrak{g}^*$ is a Lie algebra, i.e.

We have the commutator map

$$[\ ,\ ]_{\mathfrak{g}^*} : \Lambda^2 \mathfrak{g}^* \longrightarrow \mathfrak{g}^*.$$

Dually this can be expressed as a cobracket (or cocommutator) $\delta = d\pi_e$

$$\delta : \mathfrak{g} \longrightarrow \Lambda^2 \mathfrak{g}.$$

And the Jacobi identity for $[\,,\,]_{\mathfrak{g}^*}$ translates into the co-Jacobi identity for δ:

$$\mathrm{Alt}\,(\delta \otimes \mathrm{id}) \circ \delta(x) = 0 \qquad (3)$$

$$\mathrm{Alt}\,(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b.$$

<u>Def.</u> A Lie coalgebra is a vector space $\mathfrak{g}$ with a linear map $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ satisfying the co-Jacobi identity (3).

But there is also a compatibility condition.

<u>Prop.</u> $\delta([a,b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] + [\delta(a), b \otimes 1 + 1 \otimes b]$ (4)

<u>Pf.</u> Trivialize TG by right translations this will allow us to replace

Π as a function $\pi: G \to \Lambda^2 \mathfrak{g}$,

namely take $\pi(x) = \Pi(x) x^{-1}$.

then: $\Pi(xy) y^{-1} x^{-1} = x \Pi(y) y^{-1} x^{-1} + \Pi(x) y \cdot y^{-1} x^{-1}$

i.e. $\underset{\pi(xy)}{\shortparallel}$ $\qquad$ $\underset{x\,\pi(y)\,x^{-1}}{\shortparallel}$ $\qquad$ $\underset{\pi(x)}{\shortparallel}$

$$\pi(xy) = x\,\pi(y)\,x^{-1} + \pi(x).$$

or

$$\pi(xy) = \pi(x) + \mathrm{Ad}_x^{\otimes 2} \pi(y).$$

Also

$$\pi(yx) = \pi(y) + \mathrm{Ad}_y\, \pi(x).$$

Now take $x = e^{ta}$, $y = e^{tb}$

$a, b \in \mathfrak{g}$

$e^{ta} \cdot e^{tb} = e^{tb} \cdot e^{ta} \cdot e^{t^2[a,b]+\cdots}$

subtract

LHS: $\pi(e^{ta} \cdot e^{tb}) - \pi(e^{tb} \cdot e^{ta})$

$$= t^2\, d\pi_e([a,b]) + o(t^3) =$$

$$t^2\, \delta([a,b]) + o(t^3)$$

RHS:

$$(1 - \mathrm{Ad}_{e^{tb}})\,\pi(e^{ta}) + (\mathrm{Ad}_{e^{ta}} - 1)\,\pi(e^{tb})$$

$\underset{-t\,\mathrm{ad}_b+\cdots}{\shortparallel}$ $\qquad$ $\underset{t\delta(a)+\cdots}{\shortparallel}$ $\qquad$ $\underset{t\cdot\mathrm{ad}_a+\cdots}{\shortparallel}$ $\qquad$ $\underset{t\delta(b)+\cdots}{\shortparallel}$

$$= t^2\left([a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)]\right) + o(t^3)$$

Def. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a Lie coalgebra structure δ which satisfies the compatibility condition (4).

Clearly, Lie bialgebras form a category $LBA_{\mathbb{R}}$ (or $LBA_{\mathbb{C}}$).

Prop. The assignment (or any ground field) $G \mapsto (\mathrm{Lie}\, G, \delta)$ is a functor
$$PLG_k \longrightarrow LBA^{f.d.}_k, \quad k = \mathbb{R}, \mathbb{C}.$$

Examples 1) Trivial Poisson-Lie structure, $\{,\} = 0$ on any G. Then $\delta = 0$.

2) 2-dimensional Lie bialgebras

$\mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}$ basis $\underset{x}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$, $\underset{y}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

$[x,y] = y$

What are Lie bialgebra structures?

$\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g} = \langle x \wedge y \rangle$ $\alpha, \beta \in k$

$\delta(x) = \alpha\, x \wedge y$, $\delta(y) = \beta\, x \wedge y$

Co-Jacobi holds since $\Lambda^3 \mathfrak{g} = 0$.

Compatibility condition also vacuous (easy exercise).

Get Lie bialgebras $\mathfrak{b}_{\alpha,\beta}$.

<u>Exer:</u> $\mathfrak{b}_{\alpha,\beta} \cong \mathfrak{b}_{0,\beta}$ if $\beta \neq 0$

and $\mathfrak{b}_{\alpha,0} \cong \mathfrak{b}_{1,0}$ for $\alpha \neq 0$.

(symmetries $x \longmapsto x + \lambda y$
$y \longmapsto \mu y$.

3) $\mathfrak{g} = sl_2$ basis $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$.

$\delta(e)=\frac{1}{2}e\wedge h$, $\delta(f)=\frac{1}{2}f\wedge h$

$\delta(h)=0$ – standard structure corr. to quantum groups.

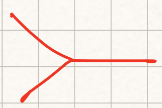
Can replace $\frac{1}{2}$ by a scalar β. Gives a 1-parameter family of non-equivalent structures. Subalgebra $\mathfrak{b}=\langle h,e\rangle$, $\langle h,f\rangle$ is a subbialgebra isomorphic to $\mathfrak{b}_{0,\pm 2\beta}$

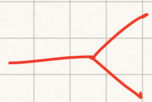
Proposition: If $(\mathfrak{g},[\,,\,],\delta)$ is a finite dimensional Lie bialgebra then $(\mathfrak{g}^*,\delta^*,[\,,\,]^*)$ is a Lie bialgebra. (I.e. category of f.d. Lie bialgebras is antiequivalent to itself).

Pr

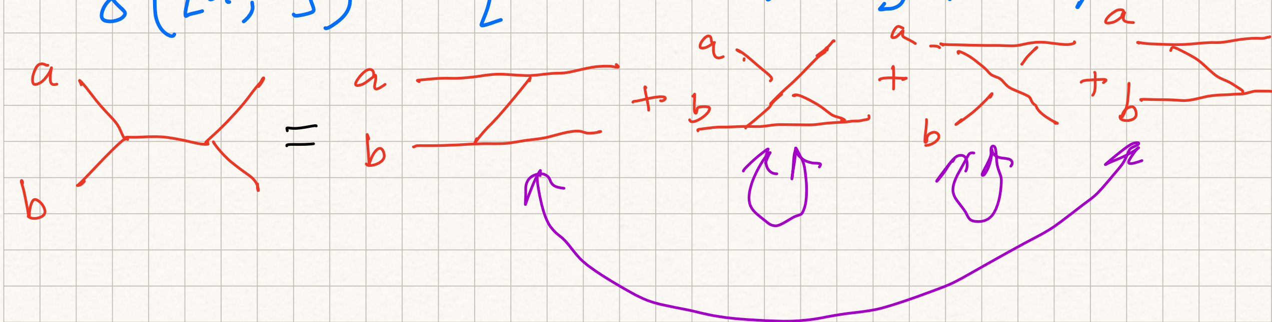
Pf. Jacobi $\Longleftrightarrow$ co-Jacobi

The self-duality of compatibility condition:

 $[\ ,\ ]$

 δ

$$\delta([a,b]) = [a\otimes 1 + 1\otimes a, \delta(b)] + [\delta(a), b\otimes 1 + 1\otimes b]$$



Main theorem of Poisson-Lie theory.

Thm. (Lie) The functor $G \mapsto \mathrm{Lie}\, G$ is an equivalence between the category of simply connected Lie groups (over $\mathbb{R}$ or $\mathbb{C}$) and the category of f.d. Lie algebras

Thm. (Drinfeld). The functor

$G \longmapsto (\mathrm{Lie}\, G, \delta)$ is an equivalence

$\mathrm{PLG}_k^{\text{s. conn.}} \xrightarrow{\sim} \mathrm{LBA}_k^{\text{f.d.}}$ ← f.d. Lie bialgebras

Also true for formal groups in char 0 but not for algebraic group.

Pf. Need to show this functor is essentially surjective and fully faithful.

Full faithfulness is easy:

$$\mathrm{Hom}_{\mathrm{PLG}}(G_1, G_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LBA}}(\mathfrak{g}_1, \mathfrak{g}_2)$$

$(\mathfrak{g}_i = \mathrm{Lie}\, G_i)$

(Exercise).

Most nontrivial part:

Every f.d. Lie bialgebra $(\mathfrak{g}, [\,,], \delta)$ comes from a Poisson-Lie group.

By fund thm of Lie theory, we already have a simply connected Lie group G corr. to $\mathfrak{g}$, so we just need to put on G a Poisson-Lie structure. So we want to define π (or Π) on G s.t. $d\pi_e = \delta$.

Cohomological interpretation:

V a G-module

$H^1(G,V) = \mathrm{Ext}^1_G(k,V)$ $\left(\begin{array}{c} k=\mathbb{R} \text{ or } \mathbb{C} \\ \text{ground field} \end{array}\right)$

classifies short exact sequences (extensions)

$$0 \to V \to W \to k \to 0$$

So $W = V \oplus k$ as a vector space

$$\rho_W(g) = \begin{pmatrix} \rho_V(g) & \pi(g) \\ 0 & 1 \end{pmatrix} \qquad \pi : G \to V$$

This is a repr ($\rho_W(xy) = \rho_W(x)\rho_W(y)$)

if and only if

$$\pi(xy) = \pi(x) + \rho_W(x)\pi(y) \qquad (5)$$

<u>Def.</u> A function $\pi : G \to V$ such that (5) holds is called a 1-cocycle of G with coefficients in V. The space of such cocycles is denoted by $Z^1(G,V)$.

<u>Example:</u> 1-coboundaries. $\pi = dv$

$$\pi(x) = v - xv, \quad v \in V.$$

$$\pi(xy) = v - xyv = v - xv + x(v - yv)$$

$$= \pi(x) + x\pi(y).$$

The space of 1-coboundaries is denoted $B^1(G,V) \subset Z^1(G,V)$.

The quotient $H^1(G,V) = \dfrac{Z^1(G,V)}{B^1(G,V)}$ is called the first cohomology

group of G with coefficients in V.
It is easy to show that two 1-cocycles define the same extension (short exact sequence) up to isomorphism $\iff$ they differ by a coboundary: $\pi_1 - \pi_2 = dv$. Thus $H^1(G,V)$ classifies such extensions.

Remark. If G is a topological group ($\mathbb{R}$- or $\mathbb{C}$-Lie group), one can require cocycles to be continuous (smooth, holomorphic). Then the corresponding H^1 will classify extensions in the relevant category.

V cont. represent. (smooth, holomorphic)

There is a similar story for Lie algebras.

$$Z^1(\mathfrak{g},V) = \{\delta : \mathfrak{g} \to V \mid \delta([a,b]) = a\delta(b) - b\delta(a)\}$$

$$B^1(\mathfrak{g},V) = \{\delta \mid \delta(a) = av,\ v\in V\}.$$

$$H^1(\mathfrak{g},V) = Z^1(\mathfrak{g},V)/B^1(\mathfrak{g},V).$$

Thus we see that the compatibility condition for Lie bialgebras is just the condition that δ is a 1-cocycle:

$$\delta \in Z^1(\mathfrak{g}, \Lambda^2\mathfrak{g}).$$

(here we care about actual cocycle, not just cohomology).

Similarly, the condition for Poisson-Lie structure is

$$\pi(xy) = \pi(x) + \mathrm{Ad}_x^{\otimes 2}\pi(y)$$

$\pi : G \to \Lambda^2\mathfrak{g}$. is a 1-cocycle.

So our job is just to show that a 1-cocycle for Lie algebras can be lifted (integrated) to a 1-cocycle for Lie groups.

As with any integration problem, we have to solve some differential equations. But it turns out that they have already been solved for us by Lie: we can intepret this problem as a problem of lifting homomorphisms which is possible by the fundamental theorems of Lie theory.

For this use the notion of semidirect product.

$V \rtimes G = V \times G$ as a manifold (set) but with twisted product:

$$(v_1, g_1) \cdot (v_2, g_2) = (v_1 + g_1 v_2, g_1 g_2)$$

Similarly for Lie algebras:

$\mathrm{Lie}(V \rtimes G) = V \oplus \mathfrak{g}$ as a space but bracket is twisted:

$$[(v_1, a_1), (v_2, a_2)] = (a_1 v_2 - a_2 v_1, [a_1, a_2])$$

Prop. (1) $\pi : G \to V$ is a 1-cocycle $\iff g \mapsto (\pi(g), g)$ is a group homomorphism $G \to V \rtimes G$.

(2) $\delta : \mathfrak{g} \to V$ is a 1-cocycle $\iff a \mapsto (\delta(a), a)$ is a Lie algebra homomorphism $\mathfrak{g} \to V \rtimes \mathfrak{g}$.

(exercise).

Cor. Any $\delta \in Z^1(\mathfrak{g}, V)$ gives rise to $\pi \in Z^1(G, V)$ regular, with $d\pi_e = \delta$.

simply conn.

Pf. Turn δ into homom.

$\tilde{\delta}: \mathfrak{g} \longrightarrow V \rtimes \mathfrak{g}$, exponentiate it to $\tilde{\pi}: G \longrightarrow V \rtimes G$, (2nd fund thm of Lie thy) $\tilde{\pi}(g) = (\pi(g), g)$, extract $\pi(g)$.

For Lie bialgebras we take $V = \Lambda^2 \mathfrak{g}$; this shows the existence of $\pi: G \to \Lambda^2 \mathfrak{g}$, a 1-cocycle with $d\pi_e = \delta$.

But we still need to show that π defines a Poisson bracket (i.e, satisfies the Jacobi identity). (i.e. $[\Pi, \Pi] = 0$)

We'll show that $[\Pi, \Pi]$ gives rise (by right translations) to $\xi: G \to \Lambda^3 \mathfrak{g}$ which is a 1-cocycle and $d\xi_e = \mathrm{Alt}\,(\delta \otimes \mathrm{Id}) \circ \delta = 0$.
(easy)
So since we have a bijection

between regular 1-cocycles for the group and 1-cocycles for the Lie algebra, we'll obtain that $\xi = 0 \implies [\Pi, \Pi] = 0$.

So it remains to show that ξ is a 1-cocycle.

$$\{f, g\}(xy) = \{f,g\}_x(xy) + \{f,g\}_y(xy)$$

$\iff \Pi$ is a 1-cocycle.

Similarly, ξ is a 1-cocycle

$$\iff \mathrm{Alt}\,\{\{f,g\},h\}(xy) =$$

$$= \mathrm{Alt}\,\{\{f,g\}_x, h\}_x(xy) +$$

$$\mathrm{Alt}\,\{\{f,g\}_y, h\}_y(xy)$$

This is equivalent to saying that cross terms vanish:

$$\mathrm{Alt}\,\{\{f,g\}_x, h\}_y^{(xy)} +$$

$$\mathrm{Alt}\,\{\{f,g\}_y, h\}_x^{(xy)} = 0.$$

$$LHS = \sum_{i,j,r,s} \Pi_{ij}(x)\Pi_{rs}(y) \times$$

$$\Big(\mathrm{Alt}\big(\partial_{r,y}\partial_{i,x}f \cdot \partial_{j,x}g \cdot \partial_{s,y}h\big)$$

$$+ \mathrm{Alt}\big(\partial_{i,x}f \cdot \partial_{r,y}\partial_{j,x}g \cdot \partial_{s,y}h\big)$$

differ by $g \leftrightarrow h$

differ by $f \leftrightarrow h$, $i \leftrightarrow j$, $r \leftrightarrow s$

$$+ \mathrm{Alt}\big(\partial_{i,x}\partial_{r,y}f \cdot \partial_{s,y}g \cdot \partial_{j,x}h\big)$$

$$+ \mathrm{Alt}\big(\partial_{r,y}f \cdot \partial_{i,x}\partial_{s,y}g \cdot \partial_{j,x}h\big)\Big)$$

So we have shown that π exists and proved Drinfeld's theorem.

But how to compute this π?

We have

$$\pi(xe^{ta}) = \pi(x) + \mathrm{Ad}_x\, \pi(e^{ta})$$

Differentiating:

(6) $L_a\pi(x) = \mathrm{Ad}_x\, \delta(a)$.

[L_a is the infinitesimal right translation by a, i.e. the left-invariant vector field corresponding to a]

This is a system of linear inhomogeneous differential equations, and it is easy to see that it has ≤ 1 solution with initial condition $\pi(e) = 0$.

Proposition.

$$\pi(e^a) = \frac{e^{ad_a} - 1}{ad(a)} \delta(a) \stackrel{def}{=} \sum_{n=0}^{\infty} \frac{ad_a^n}{(n+1)!} \delta(a).$$

Pf. First we show that (6) actually has a solution. For this we need to check compatibility:

$$\boxed{[L_a, L_b]\pi(x)} = L_a Ad_x \delta(b)$$

$$- (b \leftrightarrow a) = \frac{d}{dt}\Big|_{t=0} Ad_x Ad_{e^{ta}} \delta(b)$$

$$- (b \leftrightarrow a) = Ad_x([a \otimes 1 + 1 \otimes a, \delta(b)])$$

$$-(b \leftrightarrow a) = Ad_x \quad \delta([a,b]) =$$

$$= \boxed{L_{[a,b]} \pi(x)}$$

Now, we know that solution exists, and it in particular satisfies

$$\frac{d}{dt}\pi(e^{ta}) = Ad_{e^{ta}}\,\delta(a) \qquad (*)$$

and is determined by this equation and the initial condition $\pi(e) = 0$.

But

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^{n+1}ad_a^n}{(n+1)!}\,\delta(a) =$$

$$\sum_{n=0}^{\infty}\frac{t^n ad_a^n}{n!}\,\delta(a) = e^{t\,ad_a}\,\delta(a) =$$

$$= \boxed{Ad_{e^{ta}}\,\delta(a)},$$

So π from the statement of proposition also satisfies $(*)$, hence gives the desired 1-cocycle. ∎

Dual Poisson Lie group:

G simply connected PLG,

$\mathfrak{g} = \mathrm{Lie}\, G$ Lie bialgebra

$\mathfrak{g}^*$ – also a Lie bialgebra

G^* – simply connected Poisson Lie group with $\mathrm{Lie}\, G^* = \mathfrak{g}^*$.

But there is no simple explicit description of G^*.

Examples of duality of Lie bialgebras

G Lie group with $\{,\} = 0$

1. $\mathfrak{g} = \mathrm{Lie}\, G$ f.d. Lie algebra, $\delta = 0$

$\Rightarrow \mathfrak{g}^*$ is abelian, $\delta_{\mathfrak{g}^*} = [,]^*$.

$G^* = \mathfrak{g}^*$, $\pi(a) = \delta(a)$.

> G^*/Γ is a PLG only if Γ fixes π

2. $(b^{0,\beta})^* = b^{0,\beta^{-1}}$, $\beta \neq 0$

$[x,y] = 0$, $\delta(x) = 0$, $\delta(y) = \beta x \wedge y$

$(b^{1,0})^* = b^{1,0}$

$[x,y] = y$, $\delta(x) = x \wedge y$, $\delta(y) = 0$

3. Standard structure on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$(G = SL_2(\mathbb{C}))$ basis e, f, h of $\mathfrak{g}$

Dual basis e^*, f^*, h^* of $\mathfrak{g}^*$

$[h^*, e^*] = -2e^*$, $[h^*, f^*] = -2f^*$,

$[e^*, f^*] = 0$

$\delta(e^*) = \frac{1}{2} h^* \wedge e^*$ $\quad \delta(h^*) = 0$

$\delta(f^*) = \frac{1}{2} h^* \wedge f^*$

So get that $\mathfrak{g}^* =$ subalgebra

in $\underset{\langle x_1, y_1\rangle}{\mathfrak{b}^{0,1}} \oplus \underset{\langle x_2, y_2\rangle}{\mathfrak{b}^{0,-1}}$

of elements $\alpha(x_1 + x_2) + \beta_1 y_1 + \beta_2 y_2$

Examples of duality of PLG.

1. 2 dim case: $\quad H = \left\{ \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} \middle| p > 0 \right\}$

Solve the diff. equation:

for $\mathfrak{b}^{0,\beta}$

$$\pi \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} = \beta p^{-1} q \, x \wedge y$$

for $\mathfrak{b}^{1,0}$

$$\pi \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} = (1 - p^{-1}) \, x \wedge y$$

2. $G = SL_2(\mathbb{C})$

$$\pi\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \frac{t}{2} e \wedge h$$

$$\pi\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \frac{t}{2} f \wedge h$$

$$\pi\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = 0$$

The rest is obtained by taking products. (exercise: compute the answer!)

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ?$$

$SL_2(\mathbb{C})^* =$

$$\left\{ \left(\begin{pmatrix} p_1 & q_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p_2 & q_2 \\ 0 & 1 \end{pmatrix} \right) \middle| \, p_1 = p_2 \right\} \quad \left(\text{univ. cover}\right)$$

$$\pi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1\right) = \frac{t}{2}(x, x) \wedge (y, 0) = \frac{t}{2} x \wedge y_1$$

$$\pi\left(1, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \frac{t}{2}(x, x) \wedge (0, y) = \frac{t}{2} x \wedge y_2$$

$$\pi\left(\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}\right) = e^t (y, 0) \wedge (0, y) =$$

$$= e^t y_1 \wedge y_2$$

Coboundary Lie bialgebras.

$\mathfrak{g}$ Lie bialgebra.

Def. A coboundary structure on $\mathfrak{g}$ is an element $r \in \Lambda^2\mathfrak{g}$ s.t. $\boxed{\delta \overset{def}{=} dr}$ $\boxed{\delta(a) = [a\otimes 1 + 1\otimes a, r]}$ is a cobracket (satisfies co-Jacobi)

A Lie bialgebra equipped with coboundary str-re is called a coboundary Lie bialgebra.

Note that r is determined uniquely up to adding an element of $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. E.g. if $\mathfrak{g}$ is semisimple, it is unique if $\exists$. (as $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$)

What is the bialgebra condition in terms of r?

Define the Classical Yang-Baxter map

$CYB: \mathfrak{g}^{\otimes 2} \longrightarrow \mathfrak{g}^{\otimes 3}$ $\quad r = \Sigma a_i \otimes b_i$

$r^{13} = a_i \otimes 1 \otimes b_i$

$$\underset{\overset{\shortparallel}{[r,r]}}{CYB(r)} = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \quad \text{in } U(\mathfrak{g})^{\otimes 3}.$$

Thm. (Drinfeld) Let $\mathfrak{g}$ be a Lie algebra and $r \in \Lambda^2 \mathfrak{g}$. Then $\delta = dr$ is a Lie bialg. structure iff

$$CYB(r) \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$$

(i.e. $\overset{[r,r]}{\shortparallel}$ is $\mathfrak{g}$-invariant).

<u>Pf.</u> Direct computation (exercise).

But we will give a different proof using Poisson-Lie groups. Namely, let r be arbitrary $\delta = dr$, and let us compute the corr. bivector Π

$\boxed{G \text{ simply conn. group.}}$

$$\Pi(e^a) = \underbrace{\frac{e^{ad_a} - 1}{ad_a} \delta(a)}_{\overset{\shortparallel}{\pi}(e^a)} \cdot e^a$$

$$= \frac{e^{ad_a} - 1}{ad_a} \cdot ad_a \overset{\delta(a)}{\overset{\shortparallel}{(r)}} \, e^a$$

$$= \left(e^{ad_a} - 1\right) r e^a$$

$$= e^a r - r e^a.$$

By analytic continuation

$$\boxed{\Pi(x) = x r - r x} \quad (**)$$

another way to see it:

$\Pi(xy) = x\Pi(y) + \Pi(x)y$

$= x(yr - ry) + (xr - rx)y$

$= xyr - rxy$

so if $(**)$ holds for x and for y then holds for xy

So $[\Pi, \Pi](x) =$

$[xr, xr] - [rx, rx]$ (as $[xr, rx] = 0$)

$= x[r, r] - [r, r]x$

So $[\Pi, \Pi] = 0 \iff$

$[r, r]$ is $\mathfrak{g}$-invariant. ∎

In particular, $\delta = dr$ is a Lie bialgebra str-re if $[r, r] = 0$.

This eqn is called the Classical Yang-Baxter equation.
solutions are called classical r matrices.

A coboundary Lie bialgebra is called triangular if $[r,r]=0$.

Can transport triangular str-res along homomorphisms:

$\phi: \mathfrak{g} \to \mathfrak{a}$ $\qquad r \in \Lambda^2\mathfrak{g}$

a triangular str-re $\Longrightarrow$ $(\phi\otimes\phi)(r) \in \Lambda^2\mathfrak{a}$ also a triang. str-re.

Classification of triangular str-res.

Prop. Let $r \in \Lambda^2\mathfrak{g}$, $[r,r]=0$. Then x^r and r^x are Poisson st-res on G

(not Poisson-Lie), left-inv and right-inv. respectively.

<u>Pf</u>: $[xr, xr] = x[r,r] = 0$.
$[rx, rx] = [r,r]x = 0$.

<u>Def</u>. A triangular str-re $r \in \Lambda^2 \mathfrak{g}$ is nondegenerate if it defines an invertible map $\mathfrak{g}^* \to \mathfrak{g}$.

$\Longleftrightarrow$ Poisson str-res xr, rx on G are symplectic.

We can reduce classif. to the nondeg. case.

<u>Prop</u>. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a solution of the CYB eqn: $[r,r] = 0$

Then $\mathfrak{g}_r^+ = \mathrm{Span}\{(\mathrm{Id} \otimes f)(r), f \in \mathfrak{g}^*\}$

$\mathfrak{g}_r^- = \mathrm{Span}\{(f\otimes \mathrm{Id})(r),\ f\in\mathfrak{g}^*\}$.

(r defines two maps $\mathfrak{g}^* \to \mathfrak{g}$
$\mathfrak{g}_r^+, \mathfrak{g}_r^-$ = images of them).

Then $\mathfrak{g}_r^+, \mathfrak{g}_r^-$ are f.d.
Lie subalgebras of $\mathfrak{g}$
(of dim = rank(r)).

Pf. Proof for $\mathfrak{g}_r^+$ ($\mathfrak{g}_r^-$ similar)

$$r = \sum_{i=1}^{n} a_i \otimes b_i$$

$\Rightarrow a_i$ basis of $\mathfrak{g}_r^+$, b_i
basis of $\mathfrak{g}_r^-$

$$CYB(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$$

$$= \sum_{i,j} \Big([a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j] \Big) = 0$$

Apply $b_i^* \otimes b_j^*$ in cpts 2,3:

$[a_i, a_j] =$ lin. comb of a_k

$\Rightarrow \mathfrak{g}_r^+$ is a Lie alg.

<u>Prop.</u> $\mathfrak{g}_r^+, \mathfrak{g}_r^-$ are Lie subcoalgebras of $\mathfrak{g}$

$$(\delta \otimes 1)(r) = \sum \delta(a_i) \otimes b_i$$

$$\| $$

$$[r^{13} + r^{23}, r^{12}] \overset{CYBE}{=\!=} [r^{13}, r^{23}]$$

$$\|$$

$$\sum a_i \otimes a_j \otimes [b_i, b_j]$$

$$\Rightarrow \delta(\mathfrak{g}_r^+) \subset \mathfrak{g}_r^+ \otimes \mathfrak{g}_r^+.$$

$\mathfrak{g}$ a Lie algebra

<u>Prop.</u> $r \in \Lambda^2 \mathfrak{g}$ defines a $[r,r]=0$ nondegenerate triangular str-re

on $\mathfrak{g}$ $\Longleftrightarrow$ $\eta = r^{-1} \in \Lambda^2 \mathfrak{g}^*$
is a nondegenerate
2-cocycle on $\mathfrak{g}$ with
values in the ground field k:

$$\eta([ab],c) + \eta([bc],a) + \eta([ca],b) = 0$$

Proof. If r is a nondegenerate triangular structure then xr is a left-invariant symplectic structure on G with symplectic form $(xr)^{-1} = x \cdot r^{-1}$. But for $\eta \in \Lambda^2 \mathfrak{g}^*$, $\omega = x\eta$ is closed if and only if η is a 2-cocycle (due to Cartan's formula for $d\omega$).

Def. A Lie algebra $\mathfrak{g}$ with a nondegenerate 2-cocycle η

is called a quasi-Frobenius Lie algebra. Ex: Frobenius LA: $\eta(a,b) = f([a,b])$, $f \in \mathfrak{g}^*$

Ex. Every even-dimensional abelian Lie algebra is quasi-Frobenius.

Ex. Let $\mathfrak{g}$ be a f.d Lie algebra, V a $\mathfrak{g}$-module, and $\gamma : \mathfrak{g} \to V$ a bijective 1-cocycle.

Then $V^* \rtimes \mathfrak{g}$ is quasi-Frobenius.

Exercise. Show that a semisimple Lie algebra cannot be quasi-Frobenius unless $\mathfrak{g} = 0$.

Ex. The Lie algebra of matrices
$$\begin{array}{c} \left(\begin{array}{cc} * & * \\ 0 & 0 \end{array} \right) \begin{array}{l} \}\, n-1 \\ \}\, 1 \end{array} \end{array} \qquad \text{dim } n(n-1)$$
is quasi-Frobenius.

Thus the classification of triangular structures is given by the following theorem.

Theorem. (Drinfeld) Triangular structures on a Lie algebra $\mathfrak{g}$ are labeled by pairs $(\mathfrak{a}, \eta)$, where $\mathfrak{a} \subset \mathfrak{g}$ is a quasi-Frobenius Lie subalgebra and $\eta : \Lambda^2\mathfrak{a} \to k$ is a nondegenerate 2-cocycle.

Namely, $r = \eta^{-1} \in \Lambda^2\mathfrak{a} \subset \Lambda^2\mathfrak{g}$ and $\mathfrak{a} = \mathfrak{g}_r^+ = \mathfrak{g}_r^-$. – support of r.

Since semisimple Lie algebras $\mathfrak{g} \neq 0$ cannot carry a nondegenerate triangular structures, we should relax the definition slightly.

Def. An element $\tilde{r} \in \mathfrak{g} \otimes \mathfrak{g}$ s.t. $\tilde{r} + \tilde{r}^{21} = T \in (S^2\mathfrak{g})^{\mathfrak{g}}$

and $CYB(\tilde{r}) = 0$ is called a quasitriangular structure.

For example, any triangular structure is quasitriangular with $T = 0$.

Also every quasitriangular structure gives a coboundary one: set $r = \tilde{r} - \frac{T}{2} \in \Lambda^2 \mathfrak{g}$, then it's easy to show that

modified CYBE

$$CYB(r) = \frac{1}{4}[T_{12}, T_{23}] \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}.$$

Conversely, if $r \in \Lambda^2 \mathfrak{g}$ is a coboundary structure with $CYB(r) = \frac{1}{4}[T_{12}, T_{23}]$ for some $T \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ then

$\tilde{r}_\pm = r \pm \frac{T}{2}$ is a quasitriangular structure. $(\tilde{r}_- = -\tilde{r}_+^{21})$

In particular, if $\tilde{r}$ is

a quasitriangular structure then $\delta = d\tilde{r} = dr$ is a Lie bialgebra structure on $\mathfrak{g}$, $\delta(a) = [a\otimes 1 + 1\otimes a, r]$ and the corresponding Poisson-Lie structure on G is

$$\Pi(x) = xr - rx = x\tilde{r} - \tilde{r}x.$$

<u>Ex.</u> 1. If $\mathfrak{g}$ is abelian then every triangular (= coboundary) structure $r \in \Lambda^2\mathfrak{g}$ gives rise to $\delta = 0$.

So if $\mathfrak{g}$ is a non-abelian Lie algebra then $(\mathfrak{g}^*, 0, [,]^*)$ is not coboundary.

2. 2-dim Lie bialgebras: $\tilde{b}^{1,0}$ is triangular but $\tilde{b}^{0,\beta}$ is not coboundary for $\beta \neq 0$. (exer.).

3. Standard structure on $\mathfrak{sl}_2$:

$\delta(e) = \frac{1}{2} e\wedge h$, $\delta(f) = \frac{1}{2} f\wedge h$,
$\delta(h)=0$ is not triangular but
is quasitriangular:

$$\tilde{r} = e\otimes f + \frac{1}{4} h\otimes h$$

$$\tilde{r} + \tilde{r}^{21} = e\otimes f + f\otimes e + \frac{1}{2} h\otimes h =$$

$T = \Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$, the Casimir tensor,
$\Omega = B^{-1}$ where B is the invariant
inner product on $\mathfrak{g}$.

The Drinfeld double construction for Lie bialgebras.

<u>Def.</u> A finite dimensional Manin triple
is a triple of finite dimensional
Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where $\mathfrak{g}$
is equipped with a nondegenerate
invariant inner product $(\,,\,)$

and $\mathfrak{g}_+, \mathfrak{g}_- \subset \mathfrak{g}$ are isotropic Lie subalgebras with $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space.

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple.

Then $(,)$ induces a nondegenerate pairing $\mathfrak{g}_+ \otimes \mathfrak{g}_- \to k$, which yields an identification $\mathfrak{g}_+^* \cong \mathfrak{g}_-$ as vector spaces.

Hence we obtain a Lie bracket on $\mathfrak{g}_+^*$ or, dually, a Lie cobracket δ on $\mathfrak{g}_+$.

<u>Prop.</u> $(\mathfrak{g}_+, [,], \delta)$ is a Lie bialgebra.

<u>Pf.</u> Need to check the 1-cocycle condition

$$\delta([a,b]) = ad_a\,\delta(b) - ad_b\,\delta(a)$$

Let $f, g \in \mathfrak{g}_+^* \cong \mathfrak{g}_-$. We have

$$(f\otimes g, \delta([a,b])) = ([f,g],[a,b]) =$$

$$= ([[f,g],a], b) = ([[f,a],g],b)$$

$$+ ([f,[g,a]],b)$$

On the other hand

$$(f\otimes g, ad_a\,\delta(b))$$

$$= ([f,a]\otimes g + f\otimes[g,a], \delta(b)) =$$

$$([[f,a]_-, g] + [f,[g,a]_-], b)$$

where for $x \in \mathfrak{g}$, $x_\pm$ is the projection of x to $\mathfrak{g}_\pm$.

Thus

$$(f\otimes g, \delta([a,b]) - ad_a\,\delta(b)) =$$

$$([[f,a]_+, g], b)$$

$$+\left([f,[g,a]_+],b\right)=$$

$$\left([f,a]_+,[g,b]_-\right)$$

$$+\left([g,a]_+,[f,b]_-\right)=$$

$$\left([f,a],[g,b]_-\right)$$

$$+\left([g,a],[f,b]_-\right)$$

$$=-\left(a,[f,[g,b]_-]+[[f,b]_-,g]\right)$$

$$=-\left(f\otimes g,\ \mathrm{ad}_b\,\delta(a)\right).\qquad\blacksquare$$

Conversely, if $(\mathfrak{a},[,],\delta)$ is a Lie bialgebra then we can construct a Manin triple.

Namely, set $\mathfrak{g}=\underset{\mathfrak{g}_+}{\underset{\shortparallel}{\mathfrak{a}}}\oplus\underset{\mathfrak{g}_-}{\underset{\shortparallel}{\mathfrak{a}^*}}$

with the natural inner product

$$\left((a_1,f_1),(a_2,f_2)\right)=f_2(a_1)+f_1(a_2).$$

We already have a bracket on $\underset{\mathfrak{g}_+}{\underset{\|}{\mathfrak{a}}}$ and on $\underset{\mathfrak{g}_-}{\underset{\|}{\mathfrak{a}^*}}$

which are isotropic for the inner product by definition. So we just need to define the bracket between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ so that the form is invariant for this bracket.

Let $a \in \mathfrak{g}_+$, $f \in \mathfrak{g}_-$. Invariance of the form requires $\forall b \in \mathfrak{g}_+, g \in \mathfrak{g}_-$

$$([a,f],b) = (f, \overbrace{[b,a]}^{= -\mathrm{ad}_a b})$$

so $[a,f]_- = -\mathrm{ad}_a^* f$

coadjoint action

Similarly

$$[a,f]_+ = \mathrm{ad}_f^* a.$$

Thus

$$[a,f] = \mathrm{ad}^*_f a - \mathrm{ad}^*_a f.$$

<u>Exer.</u> Prove the Jacobi identity for this bracket.

Thus the notions of a f.d. Lie bialgebra and Manin triple are equivalent.

Now let $\mathfrak{a} = \mathfrak{g}_+$ be a Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \underset{\mathfrak{a}^*}{\underset{\shortparallel}{\mathfrak{g}_-}})$ be the corresponding Manin triple. We will now define a Lie bialgebra structure on $\mathfrak{g}$.

Set $\delta_{\mathfrak{g}} = \delta_{\mathfrak{a}} - \delta_{\mathfrak{a}^*}$ $\quad \delta(\underset{\mathfrak{a}}{\underset{\cap}{a}}, \underset{\mathfrak{a}^*}{\underset{\cap}{f}}) = \delta_{\mathfrak{a}}(a) - \delta_{\mathfrak{a}^*}(f)$

This is clearly a Lie coalgebra structure.

We will now see that δ is a 1-cocycle (and here the minus sign is crucial).

In fact, we will show that

$\mathfrak{g}_+ \otimes \mathfrak{g}_-$

$\delta_{\mathfrak{g}} = d\tilde{r}$, where $\tilde{r} = \sum_i e_i \otimes e_i^*$,

where e_i is a basis of $\mathfrak{a} = \mathfrak{g}_+$

and e_i^* the dual basis of $\mathfrak{a}^* = \mathfrak{g}_-$;

and show that $\tilde{r}$ is

a quasitriangular structure

on $\mathfrak{g}$.

Prop. $\delta = d\tilde{r}$.

Pf. Let $a \in \mathfrak{a}$. Then

$$d\tilde{r}(a) = \sum_i \left([a, e_i] \otimes e_i^* + e_i \otimes [a, e_i^*] \right) =$$

$$\sum_i \cancel{[a, e_i] \otimes e_i^*} + e_i \otimes \left(\mathrm{ad}^*_{e_i^*} a - \cancel{\mathrm{ad}^*_a e_i^*} \right)$$

$$= \sum_i e_i \otimes \mathrm{ad}^*_{e_i^*} a \in \mathfrak{a} \otimes \mathfrak{a}$$

Now $\forall f, g \in \mathfrak{a}^*$

$$\left(f \otimes g, \sum_i e_i \otimes \mathrm{ad}^*_{e_i^*} a \right) =$$

$$\sum_i (f, e_i)(g, ad^*_{e_i^*} a) =$$

$$= (g, ad^*_f a) = ([f,g], a) =$$

$$= (f \otimes g, \delta(a)) \Rightarrow$$

$$\delta(a) = d\tilde{r}(a).$$

Similarly one shows that

$\forall f \in \mathfrak{a}^*$, $d\tilde{r}(f) = -\delta_{\mathfrak{a}^*}(f)$.

"$\mathfrak{g}_- = \mathfrak{g}_+^*$"

<u>Prop:</u> $\tilde{r} + \tilde{r}^{21} = \Omega$ is $\mathfrak{g}$-invariant and $CYB(\tilde{r}) = 0$, so $\tilde{r}$ is a quasitriangular structure on $\mathfrak{g}$.

<u>Pf.</u> $\tilde{r} + \tilde{r}^{21} = \sum_i (e_i \otimes e_i^* + e_i^* \otimes e_i) = \Omega$

so this is inverse to the invariant inner product. The second statement is checked by a direct computation (exercise).

<u>Def</u> $(\mathfrak{g}, \tilde{r})$ is called the Drinfeld double of the Lie bialg. $\mathfrak{a} = \mathfrak{g}_+$.
denote $D\mathfrak{a}$

So $\mathfrak{a}, \mathfrak{a}^{*\,op}$ are Lie sub~~bial~~algebras in $\mathfrak{g}$.
opposite cocomutator $D\mathfrak{a}$

So any f.d. Lie bialgebra embeds into a quasitriangular one with an explicit r-matrix.

In fact, the Drinfeld double is in some sense a universal construction of quasitriangular structures. Namely, we have

<u>Proposition.</u> Let $(\mathfrak{g}, r)$ be a quasitriangular Lie bialgebra.
Let $\mathfrak{g}_+^r = \langle (\mathrm{Id}\otimes f)(r) \mid f \in \mathfrak{g}^* \rangle$

and $\mathfrak{g}_-^r = \langle (f\otimes Id)(r) \mid f \in \mathfrak{g}^* \rangle$.

Images of two maps $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ defined by r.

(Lie subbialgebras of $\mathfrak{g}$).

Let $\bar{\mathfrak{g}} = D\mathfrak{g}_+^r$ be the Drinfeld double of $\mathfrak{g}_+^r$, and $(\bar{\mathfrak{g}}, \mathfrak{g}_+^r, \mathfrak{g}_-^r)$ the corresponding Manin triple.

Then there exists a unique homomorphism $\pi : \bar{\mathfrak{g}} \to \mathfrak{g}$ such that $\pi|_{\mathfrak{g}_+^r} = id$, $\pi|_{\mathfrak{g}_-^r} = id$ and $\pi^{\otimes 2}(\tilde{r}) = r$ where $\tilde{r} = \sum_i e_i \otimes e_i^*$ is the quasitriangular structure on $D\mathfrak{g}_+^r$.

In particular, $\mathfrak{g}_+^r + \mathfrak{g}_-^r = \operatorname{Im} \pi$ is a Lie subalgebra of $\mathfrak{g}$ (in fact, a subbialgebra).

Moreover, $\mathfrak{g}^r \overset{def}{=} \mathfrak{g}_+^r + \mathfrak{g}_-^r \subset \mathfrak{g}$ ← support of r

is a quotient of $D\mathfrak{g}_+^r$ as a quasitriangular Lie bialgebra.

<u>Pf.</u> We showed already that $\mathfrak{g}_+^r, \mathfrak{g}_-^r$ are Lie algebras and $\mathfrak{g}_+^{r*} \cong \mathfrak{g}_-^r$. Thus the map π is naturally defined. It is also clear that $\pi^{\otimes 2}(\tilde{r}) = r$. It remains to show that π is a homomorphism of Lie algebras, which is an exercise. (direct computation).

<u>Examples.</u> 1. $(\mathfrak{g}, r)$ triangular, nondegenerate

Then $\mathfrak{g}_+^r = \mathfrak{g}_-^r = \mathfrak{g}$, $D\mathfrak{g}_+ = (\mathfrak{g} \oplus \mathfrak{g}, [\,,\,], \delta_1 - \delta_2)$

Commutator: $[a, f] = ad_a^* f - ad_f^* a$.

Map π: $\quad \pi(x_1, x_2) = x_1 - x_2$.

2. Standard structure on $\mathfrak{sl}_2(\mathbb{C})$

$$r = e\otimes f + \frac{1}{4} h\otimes h.$$

$$\mathfrak{g}^r_+ = \langle e, h\rangle = b_+$$

$$\mathfrak{g}^r_- = \langle f, h\rangle = b_-$$

$$D\mathfrak{g}_+ = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h_1 \oplus \mathbb{C}h_2$$

$$(\cong \mathfrak{sl}_2 \oplus \mathbb{C})$$

QTR str-re $\tilde{r} = e\otimes f + \frac{1}{4} h_1 \otimes h_2$

$$\pi(e) = e,\ \pi(f) = f,\ \pi(h_1) = \pi(h_2) = h.$$

Standard Lie bialgebra structure on a simple Lie algebra

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra $/\mathbb{C}$. We will generalize the standard structure on $\mathfrak{sl}_2$ to $\mathfrak{g}$, using the above method (quotient of the Drinfeld double).

Recall basics about simple Lie algebras.

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra

$\Delta \subset \mathfrak{h}^*$ root system

$\Delta = \Delta_+ \cup \Delta_-$

$A = (a_{ij})$ Cartan matrix

Γ simple roots $(\alpha_1, \dots, \alpha_r)$

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) x, h \in \mathfrak{h} \}$

1-dim space for $\alpha \in \Delta$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

Δ_+ Δ_-

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ if $\alpha+\beta \in \Delta$, $\alpha, \beta \in \Delta$.

$(,)$ nondeg. form on $\mathfrak{h}^*$

$(\alpha, \alpha) = 2$ for long roots.

$$d_i = \frac{2}{(\alpha_i, \alpha_i)} \qquad (g_\alpha, g_\beta) = 0, \ \alpha + \beta \neq 0.$$

Now consider the Lie algebra

$$\tilde{g} = n_+ \oplus h^{(1)} \oplus h^{(2)} \oplus n_-$$

$$h^{(1)} \simeq h^{(2)} \simeq h$$

$$[h^{(1)}, h^{(2)}] = 0,$$

$g_\alpha = \langle e_\alpha \rangle$, $\alpha \in \Delta_+$

$g_{-\alpha} = \langle f_\alpha \rangle$

$[e_\alpha, f_\alpha] = h_\alpha$

$\alpha \in \Delta_+$

$$[h^{(i)}, e_\alpha] = \alpha(h)\, e_\alpha$$

$$[h^{(i)}, f_\alpha] = -\alpha(h)\, f_\alpha$$

$$[e_\alpha, f_\alpha] = \frac{1}{2}\left(h_\alpha^{(1)} + h_\alpha^{(2)}\right)$$

Thus $\tilde{g} = g \oplus h$ $\quad \forall h \in h$

$$h^{(1)} = (h, h), \qquad h^{(2)} = (h, -h)$$

Then $(\tilde{g}, b_+, b_-)$ is a Manin triple (where $b_+ = \langle h^{(1)}_{h \in h}, e_\alpha \rangle$ and $b_- = \langle h^{(2)}_{h \in h}, f_\alpha \rangle$.

The invariant inner product is $(\,,\,)_{\mathfrak{g}} - (\,,\,)_{\mathfrak{h}}$.

So we have quasitriangular structure

$$\tilde{r} = \frac{1}{2}\sum_i x_i^{(1)} \otimes x_i^{(2)} + \underbrace{\sum_{\alpha\in\Delta_+} e_\alpha \otimes f_\alpha}_{\downarrow\, b_+\otimes b_-}.$$

where x_i is an orthonormal basis of $\mathfrak{h}$.

So when we take a quotient by $\mathfrak{h}$, we obtain $\mathfrak{g}$ with quasitriang. structure

$$r = \frac{1}{2}\sum_i x_i \otimes x_i + \sum_{\alpha\in\Delta_+} e_\alpha \otimes f_\alpha,$$

<u>Exercise.</u> Show that

$$\delta(e_i) = \frac{d_i}{2} e_i \wedge h_i, \quad \delta(f_i) = \frac{d_i}{2} f_i \wedge h_i,$$
$$\delta(h_i) = 0.$$

Quantization of Lie bialgebras.

Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is a associative bialgebra:

we have the comultiplication

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$
$$\longrightarrow S^2 U(\mathfrak{g})$$

which is coassociative: defines

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

(associative product on $U(\mathfrak{g})^*$)

and is an algebra homomorphism. (+ a counit).

Namely, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$. (x is a primitive element).

Definition. A quantized universal enveloping algebra (QUE) is a flat deformation $U_\hbar(\mathfrak{g})$ of $U(\mathfrak{g})$ as a bialgebra over $k[[\hbar]]$.

This means that $U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$ as a $k[[\hbar]]$-module, with multiplication $\mu_i : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$

$$\underset{\overset{\shortparallel}{ab}}{\mu(a,b)} = \underset{\overset{\shortparallel}{a\cdot b}}{\mu_0(a,b)} + \hbar\mu_1(a,b) + \hbar^2\mu_2(a,b) + \cdots$$

(usual product in $U(\mathfrak{g})$)

and comultiplication

$$\Delta(a) = \underbrace{\Delta_0(a)}_{\text{usual comult. in } U(\mathfrak{g})} + \hbar\Delta_1(a) + \hbar^2\Delta_2(a) + \cdots,$$

$\Delta_i : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

$a \in U(\mathfrak{g})$, so that Δ is coassociative and an algebra homomorphism.

Consider

$$\delta(a) = \lim_{\hbar \to 0} \frac{\Delta(a) - \Delta^{op}(a)}{\hbar} : U(\mathfrak{g}) \to \Lambda^2 U(\mathfrak{g}).$$

<u>Proposition</u>. (Drinfeld)

$\delta(\mathfrak{g}) \subset \Lambda^2\mathfrak{g} \subset \Lambda^2 U(\mathfrak{g})$

and $\delta : \mathfrak{g} \longrightarrow \Lambda^2\mathfrak{g}$ is a Lie bialgebra structure.

Def. $(\mathfrak{g}, \delta)$ is called the quasiclassical limit of the QUE algebra $U_\hbar(\mathfrak{g})$, and $U_\hbar(\mathfrak{g})$ is called a quantization of $U(\mathfrak{g})$.

Example. Let $q = e^{\hbar/2}$.

$U_\hbar(sl_2) = \langle E, F, H \rangle$

$[H, E] = 2E, \quad [H, F] = -2F,$

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

$$\Delta H = H \otimes 1 + 1 \otimes H$$

$$\Delta E = E \otimes q^H + 1 \otimes E$$

$$\Delta F = F \otimes 1 + q^{-H} \otimes F$$

Proposition. $U_\hbar(sl_2)$ is a

quantization of the standard Lie bialgebra structure on $\mathfrak{sl}_2$.

<u>Theorem</u>. (E. – Kazhdan, 1995, conjectured by Drinfeld) Let char $k=0$.
Then every Lie bialgebra $(\mathfrak{g}, \delta)$ over k admits a quantization. Moreover, there exists a quantization functor

$$Q : LBA_{k[[\hbar]]} \longrightarrow QUE_k$$

such that the quasiclassical limit of $Q(\mathfrak{g}, \delta = \delta_0 + \hbar\delta_1 + \dots)$ is $(\mathfrak{g}, \delta_0)$, and this functor is an equivalence of categories.

Ex. $Q(\mathfrak{sl}_2, \delta_{standard}) = U_\hbar(\mathfrak{sl}_2)$