Poisson Algebras II, Non-commutative Algebras

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1. Introduction

A (commutative) algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\}: D\times D\to D$, called a *Poisson bracket*, such that

1. $\{a,b\}=-\{b,a\}$ for all $a,b\in D$ (anti-commutative),
2. $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0$ for all $a,b,c\in D$ (Jacobi identity), and
3. $\{ab,c\}=a\{b,c\}+\{a,c\}b$ for all $a,b,c\in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D,I\}\subseteq I$. We denote by $\langle a\rangle$ the Poisson ideal of D generated by the element a. Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ\subseteq P \Rightarrow I\subseteq P \quad \text{or} \quad J\subseteq P,$$

where I and J are Poisson ideals of D. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K. A K-linear map $\alpha: D\to D$ is a *Poisson derivation* of D if α is a K-derivation of D and

$$\alpha(\{a,b\})=\{\alpha(a),b\}+\{a,\alpha(b)\} \quad \text{for all } a,b\in D.$$

A set of all Poisson derivations of D is denoted by $\mathrm{PDer}_K(D)$.

2. How did we get our class of Poisson algebras $\mathcal{A}$?

Lemma. [Oh] *Let D be a Poisson algebra over a field K, $c\in K$, $u\in D$ and $\alpha,\beta\in \mathrm{PDer}_K(D)$ such that*

$$\alpha\beta=\beta\alpha \quad \text{and} \quad \{d,u\}=(\alpha+\beta)(d)u \quad \text{for all } d\in D. \tag{1}$$

Then the polynomial ring $D[x,y]$ becomes a Poisson algebra with Poisson bracket

$$\{d,y\}=\alpha(d)y,\quad \{d,x\}=\beta(d)x \quad \text{and} \quad \{y,x\}=cyx+u \quad \text{for all } d\in D. \tag{2}$$

The Poisson algebra $D[x,y]$ with Poisson bracket (2) is denoted by $(D;\alpha,\beta,c,u)$.

3. How did we construct $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A}=(K[t];\alpha,\beta,c,u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a,b\}=0$ for all $a,b\in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0=\{d,u\}=(\alpha+\beta)(d)u \quad \text{for all } d\in K[t],$$

which implies that precisely one of the three classes holds:

(Class I: $\alpha+\beta=0$ and $u=0$), (Class II: $\alpha+\beta=0$ and $u\neq 0$) or (Class III: $\alpha+\beta\neq 0$ and $u=0$).

4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha,\beta\in \mathrm{PDer}_K=\mathrm{Der}_K(K[t])=K[t]\partial_t$ such that $\alpha=f\partial_t$ and $\beta=g\partial_t$, where $f,\ g\in K[t]\backslash\{0\}$, $\partial_t=d/dt$ then*

$$\alpha\beta=\beta\alpha \quad \text{if and only if} \quad g=\frac{1}{\lambda}f \quad \text{for some} \quad \lambda\in K^\times:=K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha=f\partial_t$, $\beta=\lambda^{-1}f\partial_t$, $c\in K$, $u\in K[t]$, where $f\in K[t]$ and $\lambda\in K^\times$. Then we have the class of Poisson algebras $\mathcal{A}=K[t][x,y]=(K[t];\alpha=f\partial_t,\beta=\lambda^{-1}f\partial_t,c,u)$ with Poisson bracket defined by the rule:

$$\{t,y\}=fy,\qquad \{t,x\}=\lambda^{-1}fx \quad \text{and} \quad \{y,x\}=cyx+u. \tag{4}$$

The first class of Poisson algebras $\mathcal{A}$

The first class (Class I) of the Poisson algebras $\mathcal{A}$ has two main subclasses: Class I.1 and Class I.2. The results were indicated in these six Poisson algebras $\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_6,\mathcal{A}_7,\mathcal{A}_9$ and $\mathcal{A}_{10}$. Also, we presented some of their Poisson spectrum in diagrams, see diagram 1.

Diagram 1: The 'Poisson Algebras I' poster

The first part of the second class (Class II) of Poisson algebras $\mathcal{A}$ is presented in this poster and the next diagram shows the second class structure.

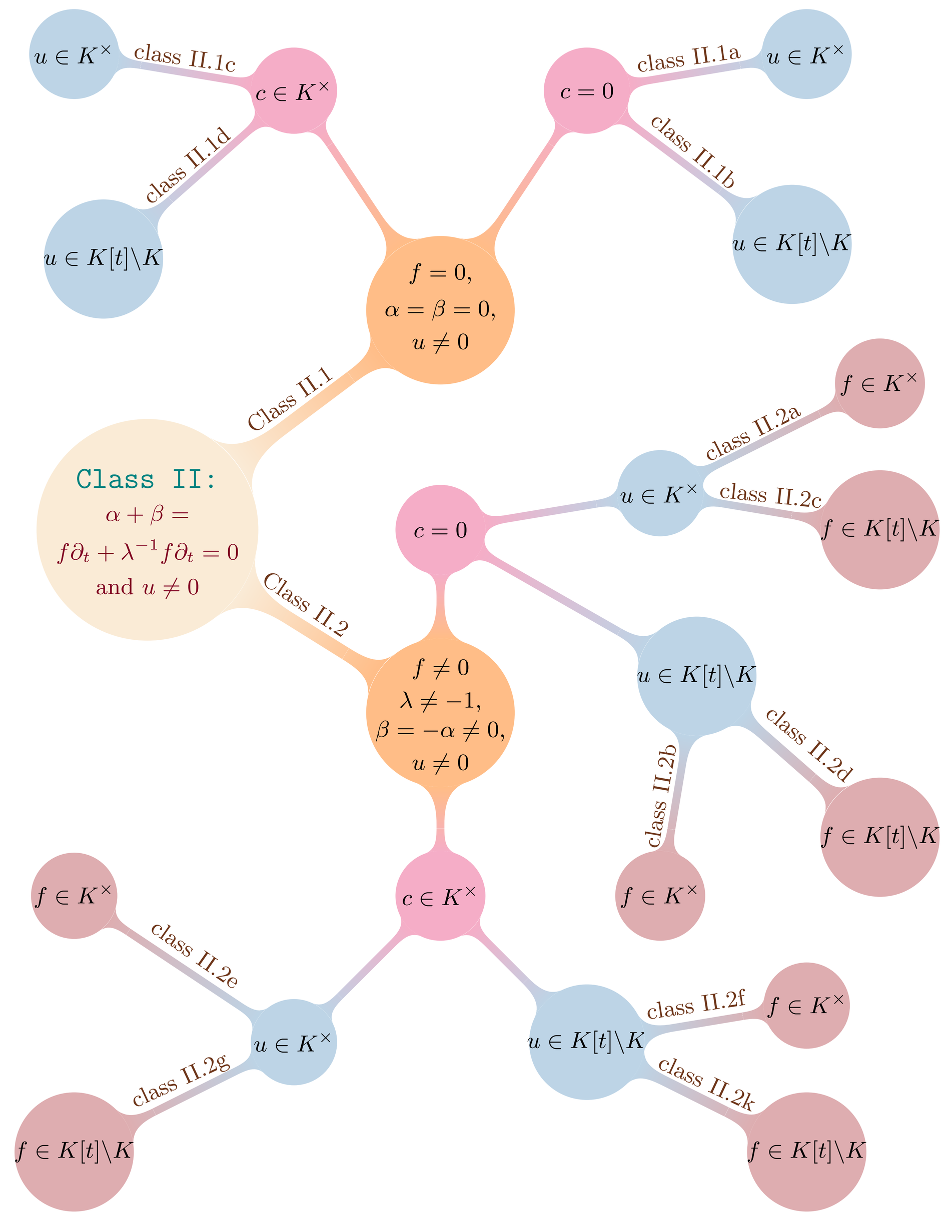

Diagram 2: Structure of the second class of Poisson algebras $\mathcal{A}$

Class II: $\alpha+\beta=f\partial_t+\frac{1}{\lambda}f\partial_t=(1+\frac{1}{\lambda})f\partial_t=0$ and $u\neq 0$

Class II.1:

If $f=0$, i.e. $\alpha=\beta=0$ and $u\in K[t]\backslash\{0\}$ then we have the Poisson algebra $\mathcal{A}_{11}=(K[t];0,0,c,u)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \quad \text{and} \quad \{y,x\}=cyx+u. \tag{5}$$

There are four subclasses.

Class II.1a:

If $c=0$ and $u\in K^\times$ then we have the Poisson algebra $\mathcal{A}_{12}=(K[t];0,0,0,u)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \quad \text{and} \quad \{y,x\}=u. \tag{6}$$

The Poisson spectrum of $\mathcal{A}_{12}$ is $\{\mathfrak{p}\otimes K[x,y]\mid \mathfrak{p}\in \mathrm{Spec}(K[t])\}$.

Class II.1b:

If $c=0$, $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\ldots,\lambda_s\}$ is the set of distinct roots of u then $\mathcal{A}_{13}=(K[t];0,0,0,u)$ is a Poisson algebra with Poisson bracket (6). The Poisson spectrum of $\mathcal{A}_{13}$ is in the below diagram, 3.

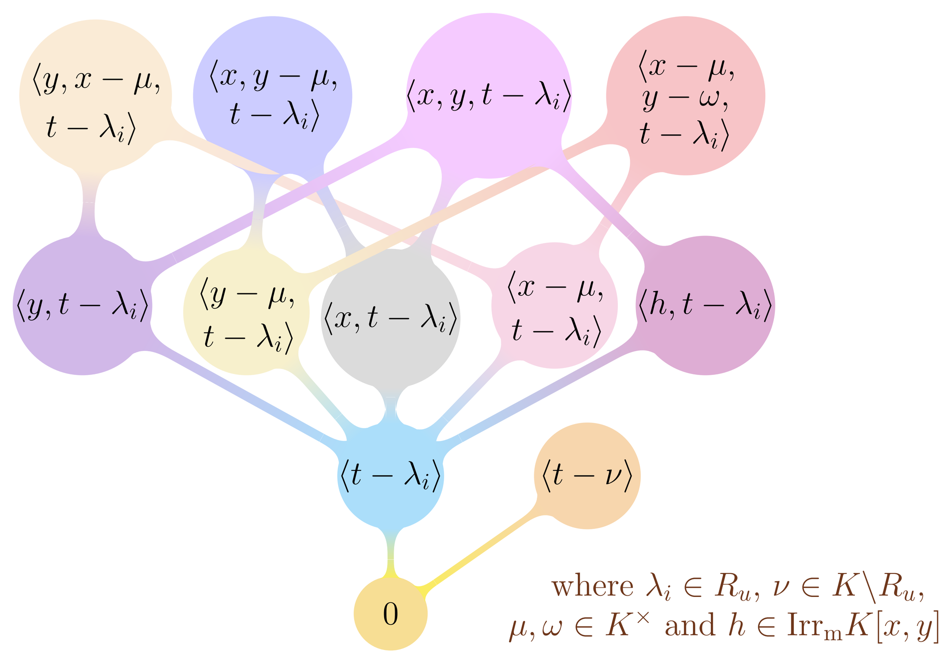

Diagram 3: The containment information between Poisson prime ideals of $\mathcal{A}_{13}$

Class II.1c:

If c and u in $K^\times$, i.e. $R_u=\emptyset$ then we have the Poisson algebra $\mathcal{A}_{14}=(K[t];0,0,c,u)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \quad \text{and} \quad \{y,x\}=cyx+u:=\rho. \tag{7}$$

The Poisson spectrum of $\mathcal{A}_{14}$ is in the below diagram, 4.

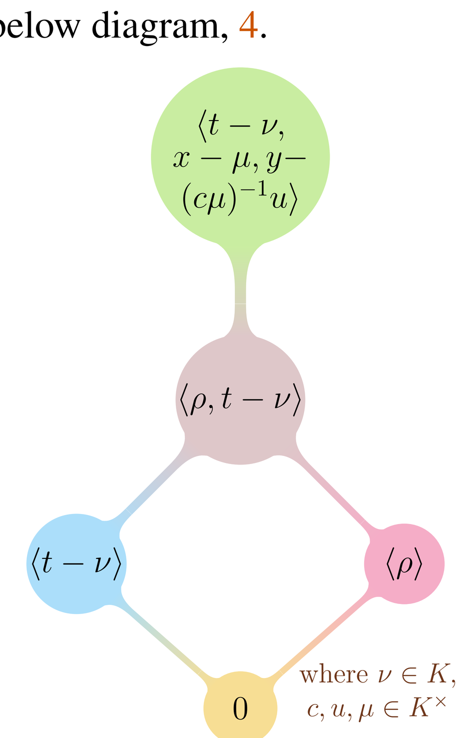

Diagram 4: The containment information between Poisson prime ideals of $\mathcal{A}_{14}$

Class II.1d:

If $c\in K^\times$, $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\ldots,\lambda_s\}$ is the set of distinct roots of u then $\mathcal{A}_{15}=(K[t];0,0,c,u)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \quad \text{and} \quad \{y,x\}=\rho. \tag{8}$$

It follows that the element $\rho=cyx+u$ is an irreducible polynomial in $\mathcal{A}_{15}$. The Poisson spectrum of $\mathcal{A}_{15}$ is in the below diagram, 5

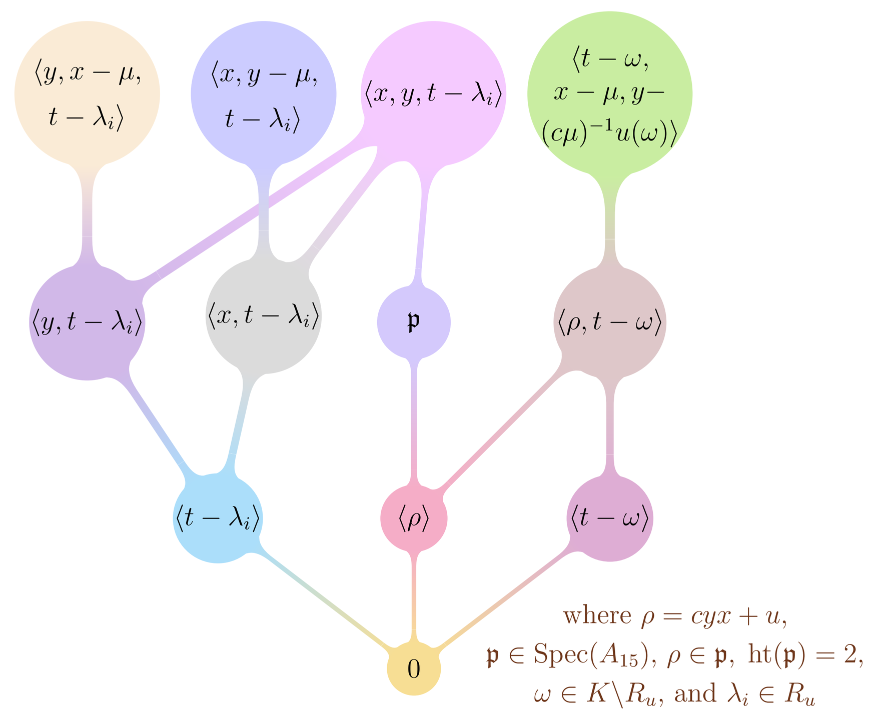

Diagram 5: The containment information between Poisson prime ideals of $\mathcal{A}_{15}$

5. Conclusion / Future research

A classification of Poisson prime ideals of Poisson algebras $\mathcal{A}$ was obtained in 12 classes out of 26. We will complete the classification. Then we aim to classify some simple finite dimensional Poisson modules over $\mathcal{A}$.

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