Polynomial Poisson algebras

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A nonempty set S' with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set S with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. S is an abelian if $g + h = h + g$.

A nonempty set V with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field K if for all $\lambda_1, \lambda_2 \in K$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
6. $1 \times v = v$.

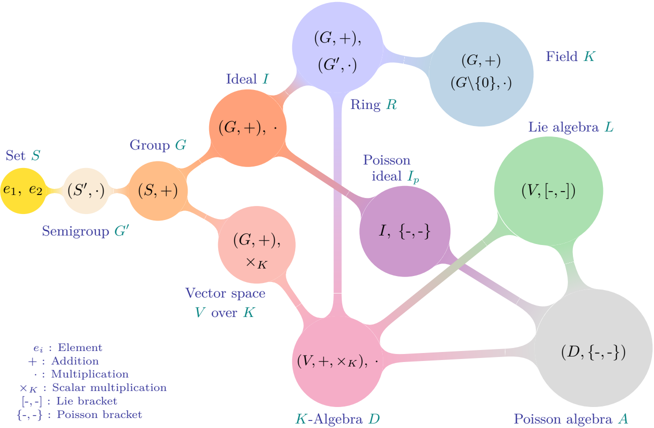

Figure 1: Algebraic structure

Outline

1. Poisson algebras
   - Poisson ideals
   - Poisson prime ideals and the Poisson spectrum
   - The Poisson centre and derivations
2. The extension of polynomial Poisson algebras
3. Examples
4. References

Poisson algebras

Definition 1

A (commutative) K-algebra $(D, +, \cdot)$ is said to be a *Poisson algebra* if there exists bilinear product $\{\text{-},\text{-}\}$ on D, called a Poisson bracket, such that $(D, \{\text{-},\text{-}\})$ is

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{a \cdot b, c\} = \{a, c\} \cdot b + a \cdot \{b, c\}$ for all $a, b, c \in D$ (Leibniz rule).

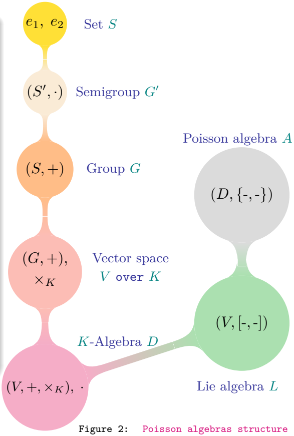

Figure 2: Poisson algebras structure

Poisson ideals

Definition 2

Let D be a Poisson algebra. A subset I of D is a *Poisson ideal* of D if

1. I is an ideal of the algebra D, and
2. $\{d, a\} \in I$ for all $d \in D$ and $a \in I$.

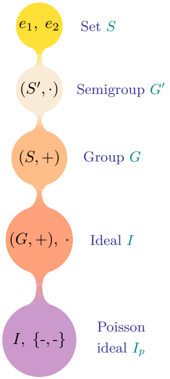

Figure 3: Poisson ideals structure

Poisson prime ideals and the Poisson spectrum

Definition 3

Let D be a Poisson algebra. A Poisson ideal P is a *Poisson prime ideal* of D if the following satisfies:

$$IJ \subseteq P \Rightarrow I \subseteq P \;\text{ or }\; J \subseteq P$$

where I and J are Poisson ideals of D.

Definition 4

Let D be a Poisson algebra. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

The Poisson centre and derivations

Definition 5

Let D be a Poisson algebra then

$$\mathrm{PZ}(D) := \{a \in D \mid \{a, d\} = 0 \text{ for all } d \in D\}$$

is called the *Poisson centre* of D.

Definition 6

Let D be an associative Poisson algebra over K. A K-linear map $\alpha : D \to D$ is said to be a *derivation* (respectively, *Poisson derivation*) on D if α satisfies 1 (respectively, satisfies 1 and 2) of the following conditions:

1. $\alpha(a \cdot b) = \alpha(a) \cdot b + a \cdot \alpha(b)$ for all $a, b \in D$;
2. $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in D$.

A set of all *derivations* (respectively, *Poisson derivations*) on D denoted by $\mathrm{Der}_K(D)$ (respectively, $\mathrm{PDer}_K(D)$).

The extension of polynomial Poisson algebras

Theorem 7 [Oh2]

Let D be a Poisson algebra over K and α, δ be K-linear maps on D. Then the polynomial ring $D[y]$ becomes a Poisson algebra with Poisson bracket:

$$\{a, y\} = \alpha(a)y + \delta(a) \quad \textit{for all } \ a \in D \tag{1}$$

iff α is a Poisson derivation on D and δ is a derivation on D such that

$$\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \delta(a)\alpha(b) - \alpha(a)\delta(b) \ \textit{ for all } \ a, b \in D. \tag{2}$$

The Poisson algebra $D[y]$ is denoted by $D[y; \alpha, \delta]$ and if δ is zero then it is denoted by $D[y; \alpha]$.

Proof:

$$(D, \{\text{-},\text{-}\}) \underset{(\alpha \in \mathrm{PDer}(D),\, \delta \in \mathrm{Der}(D))\ (2)}{\overset{\alpha,\, \delta}{\rightleftarrows}} (D[y], (1)) \qquad \boxed{D[y; \alpha, \delta]}$$

Lemma 8 [Oh2]

Let D be a Poisson algebra over K, $c \in K, u \in D$ and α, β are Poisson derivations such that

$$\alpha\beta = \beta\alpha \ \ \textit{and} \ \ \{a,u\} = (\alpha+\beta)(a)u \quad \textit{for all} \ \ a \in D \tag{3}$$

Then the polynomial ring $D[y,x]$ becomes a Poisson algebra with Poisson bracket

$$\{a,y\} = \alpha(a)y, \quad \{a,x\} = \beta(a)x \quad \textit{and} \quad \{y,x\} = cyx + u \tag{4}$$

for all $a \in D$. This Poisson algebra is denoted by $A = (D;\alpha,\beta,c,u)$ or $A = D[y;\alpha,0][x;\beta,\delta' := u\frac{d}{dy}]$.

Proof:

$$\underset{}{(D,\{\text{-},\text{-}\})} \xrightarrow[\text{By Theorem 7}]{\alpha,\ \ \delta=0} \underset{D[y;\alpha]}{(D[y],(1))} \xrightarrow[\delta' = u\frac{d}{dy}]{\beta,\ \ \beta(y)=cy} \underset{D[y;\alpha][x;\beta,\delta']}{(D[y][x],(4))} \quad \boxed{(D;\alpha,\beta,c,u)}$$

By Theorem 7 (applied at each of the two steps)

Examples

Example 9

Let $\mathfrak{gl}_n(K)$ be the set of $n \times n$ matrices over K with matrix addition and Lie bracket, i.e.

$$[A, B] = AB - BA \quad \text{for all} \quad A, B \in M_n(K).$$

Then $\mathfrak{gl}_n(K) = (M_n(K), +, [\text{-}, \text{-}])$ with Poisson bracket

$$\{A, B\} := [A, B]$$

is a Poisson algebra. Since for all $A, B, C \in \mathfrak{gl}_n(K)$

1. $\{A, B\} = AB - BA = -(BA - AB) = -\{B, A\}$.
2. $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$.
3. $\{[A, B], C\} = [A, \{B, C\}] + [\{A, C\}, B]$.

Proof 2, 3:

2. The Jacobi identity holds since

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} =$$
$$\{A, BC - CB\} + \{B, CA - AC\} + \{C, AB - BA\}$$
$$= A(BC - CB) - (BC - CB)A + B(CA - AC)$$
$$-(CA - AC)B + C(AB - BA) - (AB - BA)C$$
$$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA} + \cancel{BCA} - \cancel{BAC}$$
$$- \cancel{CAB} + \cancel{ACB} + \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC} = 0$$

3. The Leibniz rule holds since

$$[A, \{B, C\}] + [\{A, C\}, B] = \{A, \{B, C\}\} -$$
$$\{\{C, A\}, B\} = \{A, BC - CB\} - \{CA - AC, B\}$$
$$= A(BC - CB) - (BC - CB)A - (CA - AC)B$$
$$+B(CA - AC) = ABC - \cancel{ACB} - \cancel{BCA} + CBA$$
$$-CAB + \cancel{ACB} + \cancel{BCA} - BAC = ABC - BAC +$$
$$CBA - CAB = (AB - BA)C - C(AB - BA)$$
$$= \{\{A, B\}, C\} = \{[A, B], C\}$$

$$\mathfrak{s}l_n(K) = \{A \in \mathfrak{g}l_n(K) \mid \mathrm{tr}(A) = 0 \ \ (\sum(a_{ii}) = 0)\}$$

is a Poisson ideal of $\mathfrak{g}l_n(K)$. Since

1. $\mathfrak{s}l_n(K)$ is a Lie ideal of $\mathfrak{g}l_n(K)$.
2. Let $T \in \mathfrak{g}l_n(K)$ and $B \in \mathfrak{s}l_n(K)$ then $\mathrm{tr}(\{T,B\}) = \mathrm{tr}(TB - BT) = \mathrm{tr}(TB) - \mathrm{tr}(BT) = 0$, (since $\mathrm{tr}(TB) = \mathrm{tr}(BT)$), implies that $\{T,B\} \in \mathfrak{s}l_n(K)$.

Proof 1:

i) Let $A, B \in \mathfrak{sl}_n(K)$, such that $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$, then $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B) = 0$ implies that $(\mathfrak{sl}_n(K), +)$ is an abelian subgroup of $\mathfrak{gl}_n(K)$.

ii) Let $H \in \mathfrak{gl}_n(K)$ and $A \in \mathfrak{sl}_n(K)$ then $\mathrm{tr}([H,A]) = \mathrm{tr}(HA - AH) = 0$ implies that $[H,A] \in \mathfrak{sl}_n(K)$.

Example 11 [Oh2]

Let $K[y]$ be a polynomial ring. Notice that, $K[y]$ is a Poisson algebra with trivial Poisson bracket (i.e. $\{a, b\} = 0$, for all $a, b \in K[y]$). For any $f, g \in K[y]$, set

$$\alpha = f\frac{d}{dy} \quad \text{and} \quad \delta = g\frac{d}{dy}.$$

Then α is a Poisson derivation, δ is a derivation and (α, δ) satisfies (2). Hence, by Theorem 7 the algebra $K[y, x] = K[y][x; \alpha, \delta]$ is a Poisson algebra with Poisson bracket defined by the rule

$$\{y, x\} = \alpha(y)x + \delta(y) = fx + g.$$

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Thank you for listening

Further Reading

Poisson algebra applications :

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[Bav] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

Further Reading

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- Hamiltonian systems.